On the spectrum of Schrödinger operators under Riemannian coverings

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Declaration of independent work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 126/2014 on 18/11/2014.

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Summary

The spectrum of the Laplacian on a Riemannian manifold is a natural isometric invariant. However, its behavior under maps between Riemannian manifolds, which respect the geometry of the manifolds to some extent, remains largely unclear. In this thesis, we investigate the behavior of the spectrum under Riemannian coverings.

To set the stage, let $p\colon M_2 \to M_1$ be a Riemannian covering and $S_1 = \Delta + V$ a Schrödinger operator on M_1, where Δ is the (non-negative definite) Laplacian and $V\colon M_1 \to \mathbb{R}$ is smooth and bounded from below. Consider the lift $S_2 = \Delta + V \circ p$ of S_1 on M_2. It is easy to see that the bottoms (that is, the minimums) of their spectra satisfy $\lambda_0(S_1) \leq \lambda_0(S_2)$. R. Brooks was the first one to examine when the equality holds. In particular, he proved that a normal Riemannian covering of a closed manifold (that is, compact without boundary) is amenable if and only if it preserves the bottom of the spectrum of the Laplacian.

This motivated the study of the behavior of the bottom of the spectrum under amenable coverings. Generalizing former results of R. Brooks, and P. Bérard and Ph. Castillon, in a joint work with W. Ballmann and H. Matthiesen, we proved that amenable Riemannian coverings preserve the bottom of the spectrum of Schrödinger operators. In this thesis, we prove that if the covering is infinite sheeted and amenable, then $\lambda_0(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$, where λ_0^{ess} stands for the bottom of the essential spectrum. If, in addition, the manifolds are complete, we show that the spectra of the operators satisfy $\sigma(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$. As a matter of fact, we establish this result for a quite wide class of differential operators.

Although amenability is a natural assumption for the preservation of the bottom of the spectrum (in virtue of R. Brooks' result), it is not clear to what extent it is optimal. In this direction, R. Brooks, and T. Roblin and S. Tapie showed that a normal Riemannian covering preserving the bottom of the spectrum of the Laplacian is amenable, under some quite restrictive assumptions involving the spectrum of fundamental domains of the covering. In particular, these assumptions imply that the bottom of the spectrum of the Laplacian on M_1 belongs to its discrete spectrum (that is, the bottom is an isolated point of the spectrum). In a joint work with W. Ballmann and H. Matthiesen, we replaced these assumptions with some more natural, geometric assumptions. To be more precise, we showed that if the manifolds are complete, with Ricci curvature bounded from below, V and $\operatorname{grad} V$ are bounded, and $\lambda_0(S_2) = \lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$, then the covering is amenable. In this thesis, extending all the above results, we prove that if $\lambda_0(S_2) = \lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$, then the covering is amenable. It is worth to point out that we do not impose any geometric or topological assumptions on the manifolds, and the covering is not required to be normal.

Zusammenfassung

Das Spektrum des Laplace-Operators auf einer Riemannschen Mannigfaltigkeit ist eine natürliche Isometrie-Invariante. Jedoch ist das Verhalten des Spektrums unter Abbildungen zwischen Riemannschen Mannigfaltigkeiten, welche die Geometrie in gewisser Weise respektieren, weitgehend unklar. In dieser Doktorarbeit untersuchen wir das Verhalten des Spektrums unter Riemannschen Überlagerungen.

Wir betrachten folgende Situation: Sei $p\colon M_2 \to M_1$ eine Riemannsche Überlagerung und $S_1 = \Delta + V$ ein Schrödinger-Operator auf M_1, wobei Δ der (positiv-semidefinite) Laplace-Operator und $V\colon M_1 \to \mathbb{R}$ glatt und von unten beschränkt sei. Sei $S_2 = \Delta + V \circ p$ der Lift von S_1 nach M_2. Man sieht leicht, dass die Minima der Spektren die Ungleichung $\lambda_0(S_1) \leq \lambda_0(S_2)$ erfüllen. R. Brooks hat als Erster untersucht, wann die Gleichheit gilt. Er bewies insbesondere, dass eine normale Riemannsche Überlagerung einer geschlossenen (d.h. kompakten randlosen) Mannigfaltigkeit genau dann amenabel ist, wenn sie das Minimum des Spektrums des Laplace-Operators unverändert lässt.

Dies motivierte die Untersuchung des Verhaltens des Minimums des Spektrums unter amenablen Überlagerungen in allgemeinerem Kontext. Zusammen mit W. Ballmann und H. Matthiesen bewiesen wir, dass amenable Riemannsche Überlagerungen immer das Minimum des Spektrums von Schrödinger-Operatoren erhalten; dies verallgemeinert Resultate von R. Brooks sowie von P. Bérard und Ph. Castillon. In dieser Doktorarbeit beweisen wir, dass für unendlich-blättrige amenable Überlagerungen stets $\lambda_0(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$ gilt, wobei λ_0^{ess} das Minimum des wesentlichen Spektrums bezeichnet. In dem Fall, dass die Mannigfaltigkeiten zusätzlich vollständig sind, zeigen wir, dass die Spektren der beiden Operatoren die Beziehung $\sigma(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$ erfüllen. Tatsächlich beweisen wir diese Beziehung sogar für eine deutlich größere Klasse von Differentialoperatoren.

Obwohl Amenabilität eine natürliche Bedingung für die Gleichheit der Minima der Spektren ist (laut Brooks' Ergebnis), ist es unklar, inwieweit diese Bedingung optimal ist. In dieser Richtung zeigten R. Brooks sowie T. Roblin und S. Tapie unter recht restriktiven Zusatzbedingungen an das Spektrum von Fundamentalgebieten der Überlagerung, dass eine normale Riemannsche Überlagerung, die das Minimum des Spektrums des Laplace-Operators erhält, amenabel sein muss. Die genannten Zusatzbedingungen implizieren insbesondere, dass das Minimum des Spektrums des Laplace-Operators auf M_1 zum diskreten Spektrum gehört. In einer gemeinsamen Arbeit mit W. Ballmann und H. Matthiesen ersetzten wir diese Zusatzannahmen durch gewisse natürlichere geometrische Bedingungen. Genauer zeigten wir: Wenn die beteiligten Mannigfaltigkeiten vollständig sind, ihre Ricci-Krümmung nach unten beschränkt ist, V und $\operatorname{grad} V$ beschränkt sind und $\lambda_0(S_2) = \lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$ gilt, dann ist die Überlagerung amenabel. In

dieser Doktorarbeit verallgemeinern wir alle obigen Resultate und beweisen, dass allein die Voraussetzung $\lambda_0(S_2) = \lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$ bereits die Amenabilität der Überlagerung impliziert. Man beachte, dass wir keinerlei geometrische oder topologische Bedingungen an die Mannigfaltigkeiten stellen und auch nicht die Normalität der Überlagerung voraussetzen.

Contents

Introduction

The spectrum of the Laplacian on a Riemannian manifold is a natural isometric invariant. However, its behavior under maps between Riemannian manifolds, which respect the geometry of the manifolds to some extent, remains largely unclear. In this thesis, we study the behavior of the spectrum of the Laplacian under Riemannian coverings. More generally, we are interested in the behavior of the spectrum of Schrödinger operators under Riemannian coverings, that is, operators of the form $S = \Delta + V$, where Δ is the (non-negative definite) Laplacian and V is a bounded from below, smooth function.

Let $p\colon M_2 \to M_1$ be a Riemannian covering, $S_1 = \Delta + V$ a Schrödinger operator on M_1, and $S_2 = \Delta + V \circ p$ its lift on M_2. The bottoms (that is, the minimums) of their spectra always satisfy the inequality $\lambda_0(S_1) \leq \lambda_0(S_2)$. It is natural to examine when the equality holds. Brooks [9] proved that if the base manifold is closed (that is, compact without boundary), then a normal covering p preserves the bottom of the spectrum of the Laplacian if and only if p is amenable.

This theorem motivated the study of the behavior of the bottom of the spectrum under amenable coverings. Brooks [8] proved that if the base manifold is complete, of finite topological type, without boundary and the covering is a normal and amenable, then the bottom of the spectrum of the Laplacian is preserved. Bérard and Castillon [5] extended this result by showing that if the covering is amenable and the underlying manifold is complete, with finitely generated fundamental group and without boundary, then the bottom of the spectrum of any Schrödinger operator is preserved. Recently, it was proved in [3] that the bottom of the spectrum of a Schrödinger operator is preserved under amenable coverings, without any topological or geometric assumptions.

In this thesis, we prove that if p is an infinite sheeted, amenable Riemannian covering, then $\lambda_0(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$, where λ_0^{ess} stands for the bottom of the essential spectrum. If, in addition, M_1 is complete, we show that the spectra of the operators satisfy $\sigma(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$, where σ and σ_{ess} stand for the spectrum and the essential spectrum of the operator, respectively.

As a matter of fact, we establish a quite more general result. Let $p\colon M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering of not necessarily complete manifolds with possibly empty, smooth boundary. Let $E_1 \to M_1$ be a Riemannian or Hermitian vector bundle endowed with a connection ∇, and D_1 a differential operator on E_1. Let $E_2 := p^* E_1 \to M_2$ be the pullback bundle endowed with the corresponding connection ∇ and let D_2 be the lift of D_1 on E_2. As the domain of D_1 we consider the space of compactly supported smooth sections, which (if M_1 has non-empty boundary) satisfy a boundary condition of the form $a \nabla_\nu \eta + b\eta = 0$, where ν is the inward pointing normal to the boundary and a, b are functions defined on the boundary. The domain of D_2 is the space of compactly supported smooth sections, which (if M_1 has non-empty boundary) satisfy analogous boundary conditions to the sections in the domain of D_1. We consider the operator D_i as a densely defined operator in $L^2(E_i)$, $i = 1, 2$. We prove that if D_1 is essentially self-adjoint, then the spectrum of D_1 is contained in the essential spectrum of any self-adjoint extension of D_2. Moreover, we show that if D_i is symmetric and bounded from below, $i = 1, 2$, then the bottoms of their Friedrichs extensions satisfy $\lambda_0^{\text{ess}}(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$.

Although amenability of the covering is a natural assumption for the preservation of the bottom of the spectrum of Schrödinger operators (in virtue of Brooks' result [9]), it is not clear to what extend it is optimal. In this direction, Brooks [8], and Roblin and Tapie [21], proved that under some quite restrictive assumptions, if the bottom of the spectrum of the Laplacian is preserved, then the covering is amenable. These assumptions involve the spectrum of fundamental domains of the covering, which may be hard to pin down in general, and in particular, imply that the bottom of the spectrum of the Laplacian on M_1 belongs to its discrete spectrum (that is, the bottom is an isolated point of the spectrum). Moreover, in both results, the covering is assumed to be normal, with finitely generated deck transformations group. Recently, in [2], these conditions were replaced with some more natural geometric assumptions. More precisely, it was proved that if the manifolds are complete, without boundary, with Ricci curvature bounded from below, V and grad V are bounded, the bottom of the spectrum is preserved, and belongs to the discrete spectrum of S_1, then the covering is amenable. A question raised in [2] is whether the assumption on the Ricci curvature is necessary.

In this thesis, we deal with this question and establish a generalization of all the above results. Initially, using the result of [2], we prove an analogue of Brooks' result [9], involving the bottom of the Neumann spectrum of the Laplacian on manifolds with smooth boundary. Namely, we prove that if M_1 is compact with boundary, then the covering p is amenable if and only if $\lambda_0^N(M_2) = 0$, where λ_0^N stands for the bottom of the Neumann spectrum of the Laplacian. It is worth to point out that this is the first result providing amenability of a covering of manifolds with boundary. This turns out to play an important role in the study of arbitrary Riemannian coverings.

Using this result, we prove that if a Riemannian covering preserves the bottom of the spectrum of a Schrödinger operator, which belongs to the discrete spectrum of the operator on the base manifold, then the covering is amenable. It is worth to point out that we do not impose any topological or geometric assumptions on the manifolds. Since the assumptions of the previous results imply that $\lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$, this result is more general than the ones of [2, 8, 21]. Examining the optimality of this assumption, we show that it cannot be replaced with $\lambda_0(S_1)$ being an eigenvalue. For sake of completeness, we establish analogous results involving the Dirichlet and the Neumann spectrum of Schrödinger operators on manifolds with boundary.

The thesis is organized as follows:

- CHAPTER 1: We briefly discuss the background material required for the rest of the thesis.
- CHAPTER 2: We establish some properties of the spectrum of Schrödinger operators.
- CHAPTER 3: We investigate the behavior of the spectrum under assumptions on the Riemannian covering. In this chapter, we mostly focus on amenable coverings. We also observe that if the deck transformations group of the covering is infinite, then the spectrum of the operator on the covering space coincides with its essential spectrum.
- CHAPTER 4: We study coverings preserving the bottom of the spectrum. In this chapter, we prove that if a Riemannian covering preserves the bottom of the spectrum of a Schrödinger operator, which belongs to the discrete spectrum of the operator on the base manifold, then the covering is amenable. To this end, we point out a slight generalization of the main result of [2]. Using this, we are able to establish an analogue of Brooks' result [9] involving manifolds with boundary, which plays a crucial role in the proof of the main result of the chapter.

CHAPTER 1

Preliminaries

In this chapter we introduce the notation and give a brief overview of the background material needed for the rest of the thesis.

1.1 Functional Analysis

In this section we recall some definitions and standard facts from functional analysis, which may be found for instance, in [18], [23, Appendix A] and [15].

1.1.1 Spectrum of closed operators

Let $L\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ be a closed linear operator on a separable Hilbert space $\mathcal{H}$ over a field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The *spectrum* of L is given by

$$\sigma(L) := \{\lambda \in \mathbb{F} : (L - \lambda)\colon \mathcal{D}(L) \to \mathcal{H} \text{ not bijective}\}.$$

The *essential spectrum* of L is defined as

$$\sigma_{\text{ess}}(L) := \{\lambda \in \mathbb{F} : (L - \lambda)\colon \mathcal{D}(L) \to \mathcal{H} \text{ not Fredholm}\}.$$

Recall that an operator is called *Fredholm* if its kernel is finite dimensional and its range is closed and of finite codimension. The *discrete spectrum* of L is the complement of the essential spectrum in the spectrum of L, that is,

$$\sigma_d(L) := \sigma(L) \setminus \sigma_{\text{ess}}(L).$$

The *approximate point spectrum* of L, denoted by $\sigma_{\text{ap}}(L)$, is defined as the set of all $\lambda \in \mathbb{F}$, such that there exists $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}(L)$ with $\|v_n\|_{\mathcal{H}} = 1$ and $(L - \lambda)v_n \to 0$

in $\mathcal{H}$. For $\lambda \in \mathbb{F}$, a *Weyl sequence* for L and λ is a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}(L)$, such that $\|v_n\|_{\mathcal{H}} = 1$, $v_n \rightharpoonup 0$ and $(L-\lambda)v_n \to 0$ in $\mathcal{H}$, where "$\rightharpoonup$" denotes the weak convergence in $\mathcal{H}$. The *Weyl spectrum* of L, denoted by $\sigma_W(L)$, is the set of all $\lambda \in \mathbb{F}$, such that there exists a Weyl sequence for L and λ.

The following proposition is the characterization of the spectrum of a self-adjoint operator as the set of approximate eigenvalues and the well-known Weyl's criterion for the essential spectrum.

Proposition 1.1. *If L is self-adjoint, then $\sigma_{\mathrm{ap}}(L) = \sigma(L)$, $\sigma_W(L) = \sigma_{\mathrm{ess}}(L)$ and $\sigma_d(L)$ consists of isolated eigenvalues of L of finite multiplicity.*

Since we are interested in closures of operators, we need the following elementary lemma, characterizing the approximate point spectrum and the Weyl spectrum of the closure in terms of the initial operator.

Lemma 1.2. *Assume that L is the closure of an operator $T\colon \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ and consider $\lambda \in \mathbb{F}$. Then:*
(i) *$\lambda \in \sigma_{\mathrm{ap}}(L)$ if and only if there exists $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$, such that $\|v_n\|_{\mathcal{H}} = 1$ and $(T-\lambda)v_n \to 0$ in $\mathcal{H}$,*
(ii) *$\lambda \in \sigma_W(L)$ if and only if there exists $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$, such that $\|v_n\|_{\mathcal{H}} = 1$, $v_n \rightharpoonup 0$ and $(T-\lambda)v_n \to 0$ in $\mathcal{H}$.*

For an operator $T\colon \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ and $v \in \mathcal{D}(T) \smallsetminus \{0\}$, the *Rayleigh quotient* of v with respect to T is defined as

$$\mathcal{R}_T(v) := \frac{\langle Tv, v\rangle_{\mathcal{H}}}{\|v\|^2_{\mathcal{H}}}.$$

It is worth to point out that if T is symmetric then $\mathcal{R}_T(v)$ is a real number, for any non-zero $v \in \mathcal{D}(T)$. The spectrum of a self-adjoint operator L is contained in $\mathbb{R}$ and the *bottom* (that is, the minimum) of the spectrum and the bottom of the essential spectrum of L are denoted by $\lambda_0(L)$ and $\lambda_0^{\mathrm{ess}}(L)$, respectively. The following characterization of the bottom of the spectrum is due to Rayleigh.

Proposition 1.3. *If $L\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint, then*

$$\lambda_0(L) = \inf_{v\in\mathcal{D}(L)\smallsetminus\{0\}} \mathcal{R}_L(v).$$

If, in addition, L is the closure of an operator $T\colon \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$, then the bottom of the spectrum of L is given by

$$\lambda_0(L) = \inf_{v\in\mathcal{D}(T)\smallsetminus\{0\}} \mathcal{R}_T(v).$$

We end this subsection with the following standard proposition involving the bottom of the essential spectrum of a self-adjoint operator.

Proposition 1.4 ([12, Proposition 2.1]). *Let $L\colon \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator and consider $\lambda \in \mathbb{R}$. Then the interval $(-\infty, \lambda]$ intersects the essential spectrum of L if and only if for any $\varepsilon > 0$, there exists an infinite dimensional subspace $\mathcal{H}_\varepsilon \subset \mathcal{D}(L)$, such that $\mathcal{R}_L(v) < \lambda + \varepsilon$, for any $v \in \mathcal{H}_\varepsilon \setminus \{0\}$.*

1.1.2 Friedrichs extension

Let $T\colon \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined, symmetric linear operator. Assume that T is *bounded from below*, that is, there exists $c \in \mathbb{R}$, such that

$$\langle Tv, v\rangle_{\mathcal{H}} \geq c\|v\|^2_{\mathcal{H}}, \tag{1.1}$$

for all $v \in \mathcal{D}(T)$, or equivalently, $\mathcal{R}_T(v) \geq c$, for any non-zero $v \in \mathcal{D}(T)$. Fix a lower bound $c \in \mathbb{R}$ of T, that is, a c for which (1.1) holds, and consider the inner product

$$\langle v_1, v_2\rangle_{\mathcal{H}_1} := \langle Tv_1, v_2\rangle_{\mathcal{H}} + (1-c)\langle v_1, v_2\rangle_{\mathcal{H}}$$

on $\mathcal{D}(T)$. Let $\mathcal{H}_1$ be the completion of $\mathcal{D}(T)$ with respect to this inner product. Evidently $\mathcal{H}_1$ may be identified with a dense subspace of $\mathcal{H}$ via a continuous injection.

The domain $\mathcal{D}(T^{(F)})$ of the Friedrichs extension $T^{(F)}$ of T is defined as the space of all $v \in \mathcal{H}_1$ for which there exists $v' \in \mathcal{H}$, such that $\langle v', w\rangle_{\mathcal{H}} = \langle v, w\rangle_{\mathcal{H}_1}$, for all $w \in \mathcal{H}_1$. For $v \in \mathcal{D}(T^{(F)})$, we define $T^{(F)}v := v' + (c-1)v$. Then $T^{(F)}$ is called the *Friedrichs extension* of T and is a self-adjoint extension of T.

Proposition 1.5. *The bottom of the spectrum of the Friedrichs extension of T is given by*

$$\lambda_0(T^{(F)}) = c - 1 + \inf_{v \in \mathcal{H}' \setminus \{0\}} \frac{\|v\|^2_{\mathcal{H}_1}}{\|v\|^2_{\mathcal{H}}},$$

where the infimum may be taken over any subspace $\mathcal{H}'$, with $\mathcal{D}(T) \subset \mathcal{H}' \subset \mathcal{H}_1$.

Proof: Evidently, for a non-zero $v \in \mathcal{D}(T^{(F)})$, we have

$$c - 1 + \frac{\|v\|^2_{\mathcal{H}_1}}{\|v\|^2_{\mathcal{H}}} = \mathcal{R}_{T^{(F)}}(v).$$

From Proposition 1.3, we obtain the asserted equality, where the infimum is taken over all $v \in \mathcal{D}(T^{(F)}) \setminus \{0\}$. From the definition of $\mathcal{H}_1$, it is easy to see that we obtain the same infimum for $v \in \mathcal{D}(T) \setminus \{0\}$ and for $v \in \mathcal{H}_1 \setminus \{0\}$. ∎

In particular, the Friedrichs extension of an operator preserves its lower bound, that is, we have

$$\lambda_0(T^{(F)}) = \inf_{v \in \mathcal{D}(T) \setminus \{0\}} \mathcal{R}_T(v). \tag{1.2}$$

1.2 Riemannian coverings

Throughout this thesis, manifolds are assumed to be connected with not necessarily connected, possibly empty, smooth boundary, unless otherwise stated. In particular, Riemannian coverings are assumed to be between connected manifolds, unless otherwise stated. For reasons that become clear in Section 4.3, we consider possibly non-connected covering spaces at some points.

A map $p\colon M_2 \to M_1$ between Riemannian manifolds, with M_2 non-connected, is called a *Riemannian covering* if the restriction of p on each connected component of M_2 is a Riemannian covering over M_1, and any point of M_1 has an open neighborhood that is evenly covered with respect to all of these restrictions.

Let $p\colon M_2 \to M_1$ be a Riemannian covering of (connected and) complete manifolds without boundary. For $x \in M_1$ and $y \in p^{-1}(x)$, the *fundamental domain* of p centered at y is defined by

$$D_y := \{z \in M_2 : d(z,y) \leq d(z,y') \text{ for all } y' \in p^{-1}(x)\}.$$

Some basic properties of these fundamental domains are presented in [3]. It is clear that D_y is closed and M_2 is the union of D_y, with $y \in p^{-1}(x)$. Moreover, ∂D_y and the cut locus $\mathrm{Cut}(x)$ of x are of measure zero, and $p\colon D_y \smallsetminus \partial D_y \to M_1 \smallsetminus C_0$ is an isometry, where C_0 is a subset of $\mathrm{Cut}(x)$. In the following lemmas and in the sequel, we denote open and closed balls by B and C, respectively.

Lemma 1.6 ([3, Lemma 2.3]). *If $K \subset B(x,r)$, then $p^{-1}(K) \cap D_y \subset B(y,r)$. In particular, if K is compact, then $p^{-1}(K) \cap D_y$ is compact.*

Proof: Let $z \in p^{-1}(K) \cap D_y$ and consider a minimizing geodesic γ from $p(z)$ to x. Since $K \subset B(x,r)$, it is clear that $\ell(\gamma) < r$, where $\ell(\cdot)$ stands for the length of a curve. Let $\tilde{\gamma}$ be the lift of γ starting from z, and let y' be its endpoint. Since $z \in D_y$, it is clear that

$$d(z,y) \leq d(z,y') \leq \ell(\tilde{\gamma}) = \ell(\gamma) < r,$$

which proves the asserted claim. ∎

Lemma 1.7 ([3, Lemma 2.2]). *For any $r > 0$, there exists $N(r) \in \mathbb{N}$, such that any $z \in M_2$ is contained in at most $N(r)$ of the balls $C(y,r)$, with $y \in p^{-1}(x)$.*

Proof: Let $z \in M_2$ and assume that it lies in the intersection of N_z pairwise different balls $C(y_i,r)$, $i = 1,\dots,N_z$. Let $\gamma_i\colon [0,1] \to M_2$ be a minimizing geodesic from y_i to z and consider the path $\sigma_i = p \circ \gamma_i$ from x to $p(z)$, $i = 1,\dots,N_z$. Since y_i's are pairwise different, it follows that the concatenations $\sigma_i \star \sigma_1^{-1}$ are pairwise non-homotopic and have length at most $2r$. Let $p_1\colon \tilde{M} \to M_1$ be the universal covering of M_1 and fix $u \in p_1^{-1}(x)$. It is clear that

$$N_z \leq \#\{w \in p_1^{-1}(x) : d(w,u) \leq 2r\},$$

where the latter one is finite, since $p_1^{-1}(x)$ is discrete. ∎

Lemma 1.8. *Consider the universal coverings $p_i\colon \tilde{M} \to M_i$, $i = 1, 2$. For any r, $r_0 > 0$, there exists $\tilde{N}(r, r_0) \in \mathbb{N}$, such that*

$$\#\{w \in p_2^{-1}(z) : B(w, r_0) \cap C(u, r) \neq \emptyset\} \leq \tilde{N}(r, r_0),$$

for any $u \in p_1^{-1}(x)$ and $z \in M_2$.

Proof: The proof is similar to the one of Lemma 1.7. ∎

1.2.1 Amenable coverings

In this subsection, we present the definition and some basic properties of amenable coverings. A right action of a countable group Γ on a countable set X is called *amenable* if there exists a Γ-invariant mean on $L^\infty(X)$. A countable group Γ is called *amenable* if the right action of Γ on itself is amenable.

Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_2 possibly non-connected. Fix $x \in M_1^\circ$ and consider the fundamental group $\pi_1(M_1)$ of M_1 with base point x. For $g \in \pi_1(M_1)$, let $\gamma_g\colon [0, 1] \to M_1$ be a representative loop. For $y \in p^{-1}(x)$, lift γ_g to a path $\tilde{\gamma}_g$, with $\tilde{\gamma}_g(0) = y$ and set $y \cdot g := \tilde{\gamma}_g(1)$. In this way, we obtain a right action of $\pi_1(M_1)$ on $p^{-1}(x)$. The covering p is called *amenable* if this right action is amenable.

This definition coincides with the definition presented in [2,3,20] in terms of the right cosets of $\pi_1(M_2)$ in $\pi_1(M_1)$, when M_2 is connected. However, this definition allows us to extend the notion of amenable coverings in case M_2 is non-connected.

For instance, consider a Riemannian covering $p\colon M_2 \to M_1$, where M_2 has countably many connected components $M_2^{(n)}$, $n \in \mathbb{N}$. If, for some $n \in \mathbb{N}$, the restriction $p\colon M_2^{(n)} \to M_1$ is amenable, then the covering $p\colon M_2 \to M_1$ is amenable. Indeed, if there exists a $\pi_1(M_1)$-invariant mean μ_n on $L^\infty(p^{-1}(x) \cap M_2^{(n)})$, for some $n \in \mathbb{N}$, then the linear functional $\mu\colon L^\infty(p^{-1}(x)) \to \mathbb{R}$, defined by

$$\mu(f) := \mu_n(f|_{p^{-1}(x) \cap M_2^{(n)}}),$$

for any $f \in L^\infty(p^{-1}(x))$, is a $\pi_1(M_1)$-invariant mean on $L^\infty(p^{-1}(x))$. However, the covering $p\colon M_2 \to M_1$ may be amenable, even when the restriction $p\colon M_2^{(n)} \to M_1$ is non-amenable, for any $n \in \mathbb{N}$.

The following characterization of amenable actions is due to Følner (cf. [5, Section 2]).

Proposition 1.9. *The right action of a countable group Γ on a non-empty, countable set X is amenable if and only if for any finite $G \subset \Gamma$ and $\varepsilon > 0$, there exists a non-empty, finite $F \subset X$, such that*

$$\#(F \setminus Fg) < \varepsilon \#(F),$$

for all $g \in G$. Such a set F is called a Følner set *for G and ε.*

In particular, a Riemannian covering $p\colon M_2 \to M_1$, with M_2 possibly non-connected, is amenable if and only if right the action of any finitely generated subgroup of $\pi_1(M_1)$ on $p^{-1}(x)$ is amenable. For a smoothly bounded, compact and connected neighborhood K of x, we denote by $i_*\pi_1(K)$ the image of the fundamental group of K in $\pi_1(M_1)$. It is clear that $p\colon p^{-1}(K) \to K$ is a Riemannian covering of manifolds with boundary, where $p^{-1}(K)$ is possibly non-connected. Evidently, the covering $p\colon p^{-1}(K) \to K$ is amenable if and only if the right action of $i_*\pi_1(K)$ on $p^{-1}(x)$ is amenable.

Proposition 1.10. *The covering $p\colon M_2 \to M_1$ is amenable if and only if the covering $p\colon p^{-1}(K) \to K$ (where $p^{-1}(K)$ may be non-connected) is amenable, for any smoothly bounded, compact and connected neighborhood K of x.*

Proof: From Proposition 1.9, it suffices to prove that for any finite subset G of $\pi_1(M_1)$, there exists a smoothly bounded, compact and connected neighborhood K of x, such that $G \subset i_*\pi_1(K)$. Let G be a finite subset of $\pi_1(M_1)$ and consider a representative loop $\gamma_g\colon [0,1] \to M_1^\circ$, for each $g \in G$. Let C be the union of the images of these loops and let U be a relatively compact, open neighborhood of C. Consider $\chi \in C^\infty(M_1)$, with $0 \leq \chi \leq 1$, $\chi = 1$ in C and $\operatorname{supp} \chi \subset U \cap M_1^\circ$. From Sard's Theorem, it follows that for almost any $t \in (0,1)$, the level set $\{\chi = t\}$ is a smooth hypersurface of M_1. Consider such a t, and the smoothly bounded, compact set $K' := \{\chi \geq t\}$. Then for the connected component K of K' containing x, we have $G \subset i_*\pi_1(K)$. ∎

Let $p\colon M_2 \to M_1$ be a Riemannian covering. It is clear that if p is finite sheeted, then p is amenable. Moreover, if p is normal, then p is amenable if and only if its deck transformations group is amenable.

We end this subsection, with some elementary properties and examples of amenable groups. The following criteria for amenability of groups are immediate consequences of the definition and Proposition 1.9, and may be found for instance, in [9] and [5, Section 2].

Corollary 1.11. *Any finitely generated group of subexponential growth is amenable.*

Corollary 1.12. *A countable group Γ is amenable if and only if any finitely generated subgroup of Γ is amenable*

Corollary 1.13. *Any countable solvable group is amenable.*

Proof: From Corollaries 1.11 and 1.12, it follows that any countable abelian group is amenable. From the definition, it is clear that an extension of an amenable group by an amenable group is also amenable. ∎

Example 1.14. The free group with two generators is non-amenable.

It is worth to point out that if Γ is an amenable group then any right action of Γ on any countable set X is amenable. In particular, if M_1 is a manifold with amenable fundamental group, then any covering $p\colon M_2 \to M_1$ is amenable.

1.2.2 Coverings of manifolds with boundary

The aim of this subsection is to show the following proposition, according to which, any Riemannian covering of manifolds with boundary can be extended to a Riemannian covering of manifolds without boundary.

Proposition 1.15. *Let M be a Riemannian manifold with boundary. Then there exists a Riemannian manifold N of the same dimension, without boundary and an isometric embedding $i\colon M \to N$, such that, after identifying M with $i(M)$, any Riemannian covering $p\colon M' \to M$ can be extended to a Riemannian covering $p\colon N' \to N$.*

In order to prove this proposition, we need to establish some auxiliary lemmas.

Lemma 1.16. *Let M be a Riemannian manifold with boundary. Then there exists a Riemannian manifold N of the same dimension, without boundary, an isometric embedding $i\colon M \to N$ and a strong deformation retraction of N onto $i(M)$.*

Proof: Consider the space $\partial M \times [0, +\infty)$ and the map $\Psi\colon \partial M \to \partial M \times [0, +\infty)$, defined by $\Psi(x) := (x, 0)$. Then $N := M \cup_\Psi (\partial M \times [0, +\infty))$ is a smooth manifold and there exists a smooth embedding $i\colon M \to N$. Therefore, M can be identified with $i(M)$. Since M is connected, so is N, and there exists a strong deformation retraction of N onto M, obtained by considering $F_t(x, r) := (x, (1-t)r)$ in the glued ends $\partial M \times [0, +\infty)$.

It remains to extend the Riemannian metric of M to a Riemannian metric of N. Any $x \in \partial M$ has an open neighborhood U_x in N, such that there exists a smooth frame field $\{e_1, \dots, e_m\}$ in U_x, where m is the dimension of the manifolds. Let $g_{jk} := \langle e_j, e_k \rangle$, $1 \leq j, k \leq m$, be the components of the Riemannian metric of M. Since they are smooth up to the boundary of M, they can be extended smoothly to a neighborhood of x. After passing to a smaller neighborhood of x if needed, we may assume that g_{jk}'s are smooth in U_x and their matrix is symmetric and positive definite at any point of U_x. Hence, they express a Riemannian metric in U_x.

Clearly, ∂M can be covered with such neighborhoods U_x. Consider the interior of M as an open subset of N endowed with its Riemannian metric and $N \smallsetminus M$ with an arbitrary Riemannian metric. Combining these Riemannian metrics via a partition of unity subordinate to this open cover of N, gives rise to a Riemannian metric of N, which is an extension of the Riemannian metric of M. ∎

Lemma 1.17. *Let M be a Riemannian manifold with boundary. Consider N as in the previous lemma and identify M with $i(M)$. Let $q\colon \tilde{N} \to N$ be the universal covering of N. Then the restriction $q\colon q^{-1}(M) \to M$ is the universal covering of M.*

Proof: Since there exists a strong deformation retraction of N onto M, every loop in N can be homotoped to a loop in M. This implies that for any $x \in M$ and $y_1, y_2 \in q^{-1}(x)$, there exists a path in $q^{-1}(M)$ from y_1 to y_2. Since M is connected, it follows that so is $q^{-1}(M)$ and the restriction $q\colon q^{-1}(M) \to M$ is a covering of (connected) manifolds.

Let $r_M\colon N \to M$ be a retraction. Then the map $r_M \circ q\colon \tilde{N} \to M$ is continuous and $r_M \circ q = q$ in $q^{-1}(M)$. From the Lifting Theorem, it has a continuous lift $\tilde{r}_M\colon \tilde{N} \to q^{-1}(M)$, with $\tilde{r}_M(y_0) = y_0$, for some $y_0 \in q^{-1}(M)$. Since $\tilde{r}_M|_{q^{-1}(M)}$ has a fixed point and is a deck transformation of the covering $q\colon q^{-1}(M) \to M$, it follows that $\tilde{r}_M\colon \tilde{N} \to q^{-1}(M)$ is a retraction. Since $\tilde{N}$ is simply connected, this yields that so is $q^{-1}(M)$. ∎

Proof of Proposition 1.15: Consider N and $q\colon \tilde{N} \to N$ as in the above lemmas, identify M with $i(M)$ and set $\tilde{M} := q^{-1}(M)$. Denote by Γ_N and Γ_M the deck transformations groups of $q\colon \tilde{N} \to N$ and $q\colon \tilde{M} \to M$, respectively. It is clear that for $g \in \Gamma_N$, we have $g|_{\tilde{M}} \in \Gamma_M$, and any $\gamma \in \Gamma_M$ has a unique extension $\gamma' \in \Gamma_N$. For any Riemannian covering $p\colon M' \to M$, there exists a subgroup $\Gamma \subset \Gamma_M$, such that $M' = \tilde{M}/\Gamma$. For $\Gamma' := \{\gamma' \in \Gamma_N : \gamma \in \Gamma\}$ and $N' := \tilde{N}/\Gamma'$, the inclusion $\tilde{M} \hookrightarrow \tilde{N}$ descends to an isometric embedding $M' \to N'$, which completes the proof. ∎

1.2.3 Lifts of differential operators

Let $p\colon M_2 \to M_1$ be a Riemannian covering of m-dimensional manifolds, $E_1 \to M_1$ a Riemannian or Hermitian vector bundle of rank κ and $D_1\colon \Gamma(E_1) \to \Gamma(E_1)$ a differential operator of order d. Consider the pullback bundle $E_2 := p^*E_1$ on M_2, $y \in M_2$ and set $x := p(y)$. Let U_2 be an open neighborhood of y, such that the restriction $p|_{U_2}$ is an isometry onto its image U_1. The *lift* $D_2\colon \Gamma(E_2) \to \Gamma(E_2)$ of D_1 is the differential operator defined by

$$D_2\eta(z) := (p|_{U_2})^*(D_1((p|_{U_2}^{-1})^*\eta)(p(z))),$$

for any $\eta \in \Gamma(E_2)$ and $z \in U_2$. After passing to a smaller neighborhood, if needed, we may assume that U_1 is contained in a coordinate neighborhood and there exists a trivialization $E_1|_{U_1} \to U_1 \times \mathbb{F}^\kappa$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. With respect to this coordinate system and trivialization, D_1 is expressed as

$$D_1 = \sum_{|\alpha| \le d} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{1.3}$$

where A^α are smooth maps defined on U_1, with values $\kappa \times \kappa$ matrices with entries in $\mathbb{F}$. Then, with respect to the lifted coordinate system and the corresponding trivialization $E_2|_{U_2} \to U_2 \times \mathbb{F}^\kappa$, D_2 has the local expression

$$D_2 = \sum_{|\alpha| \le d} (A^\alpha \circ p) \frac{\partial^{|\alpha|}}{\partial y^\alpha}.$$

Lemma 1.18. *Let M be a Riemannian manifold, $E \to M$ a Riemannian or Hermitian vector bundle endowed with a connection ∇ and $D\colon \Gamma(E) \to \Gamma(E)$ a differential operator. If M has empty boundary, set $\mathcal{D}(D) := \Gamma_c(E)$. If M has non-empty boundary, let a, b be real or complex valued functions (depending on whether E is Riemannian or Hermitian) defined on ∂M, let ν be the inward pointing normal to ∂M and consider*

$$\mathcal{D}(D) := \{\eta \in \Gamma_c(E) : a\nabla_\nu \eta + b\eta = 0 \text{ on } \partial M\}.$$

Then the operator $D\colon \mathcal{D}(D) \subset L^2(E) \to L^2(E)$ is closable.

Proof: Consider the formal adjoint D^{ad} of D, defined by

$$\langle D\eta, \theta \rangle = \langle \eta, D^{\mathrm{ad}}\theta \rangle,$$

for all $\eta \in \mathcal{D}(D)$ and $\theta \in \Gamma_{cc}(E)$, where $\Gamma_{cc}(E)$ is the space of smooth sections, compactly supported in the interior of M. It is clear that the operator

$$D^{\mathrm{ad}}\colon \Gamma_{cc}(E) \subset L^2(E) \to L^2(E)$$

is densely defined and its adjoint satisfies $D \subset (D^{\mathrm{ad}})^*$. Since the adjoint is closed, it follows that D is closable. ∎

CHAPTER 2

Schrödinger operators

Recall that throughout this thesis, manifolds are assumed to be connected with (possibly empty) not necessarily connected, smooth boundary, unless otherwise stated. Let M be a possibly non-connected Riemannian manifold. A *Schrödinger operator* on M is an operator of the form $S = \Delta + V$, where Δ is the (non-negative definite) Laplacian and $V\colon M \to \mathbb{R}$ is smooth and bounded from below. On the space $C_c^\infty(M)$ consider the inner product

$$\langle f, g\rangle_{H_V(M)} := \int_M (\langle \operatorname{grad} f, \operatorname{grad} g\rangle + (V - \inf_M V + 1) f g).$$

If M has empty boundary, let $H_V(M)$ be the completion of $C_c^\infty(M)$ with respect to this inner product. If M has non-empty boundary, let $H_V(M)$ be the completion of $\{f \in C_c^\infty(M) : \nu(f) = 0 \text{ on } \partial M\}$ with respect to this inner product, where ν is the inward pointing normal to ∂M. It is clear that $H_V(M)$ can be identified with a dense subspace of $L^2(M)$ via a continuous injection.

If M has empty boundary, we are interested in the Friedrichs extension of the operator

$$S\colon C_c^\infty(M) \subset L^2(M) \to L^2(M). \tag{2.1}$$

If M has non-empty boundary, we are interested in the Neumann extension of S, that is, the Friedrichs extension of

$$S\colon \{f \in C_c^\infty(M) : \nu(f) = 0 \text{ on } \partial M\} \subset L^2(M) \to L^2(M). \tag{2.2}$$

In any of these cases, we denote this Friedrichs extension by S^N and its domain by $\mathcal{D}(S^N)$. It is worth to point out that the space $H_V(M)$ plays the role of $\mathcal{H}_1$ in the discussion of the Friedrichs extension in Subsection 1.1.2 (where we consider the lower bound $c := \inf_M V$ for the operator).

The spectrum and the essential spectrum of S^N are denoted by $\sigma^N(S)$ and $\sigma^N_{\mathrm{ess}}(S)$, respectively, and their bottoms (that is, their minimums) by $\lambda_0^N(S)$ and $\lambda_0^{N,\mathrm{ess}}(S)$, respectively. These sets and quantities for the Laplacian are denoted by $\sigma^N(M)$, $\sigma^N_{\mathrm{ess}}(M)$ and $\lambda_0^N(M)$, $\lambda_0^{N,\mathrm{ess}}(M)$, respectively. If M has empty boundary, we sometimes drop the superscript "N" in the notation of the spectrum, the essential spectrum and their bottoms.

If M has non-empty boundary, the Dirichlet extension S^D of S is the Friedrichs extension of the operator

$$S\colon \{f \in C_c^\infty(M) : f = 0 \text{ on } \partial M\} \subset L^2(M) \to L^2(M). \tag{2.3}$$

The spectrum and the essential spectrum of S^D are denoted by $\sigma^D(S)$ and $\sigma^D_{\mathrm{ess}}(S)$, respectively, and their bottoms by $\lambda_0^D(S)$ and $\lambda_0^{D,\mathrm{ess}}(S)$, respectively. In case of the Laplacian, we denote these sets and quantities by $\sigma^D(M)$, $\sigma^D_{\mathrm{ess}}(M)$ and $\lambda_0^D(M)$, $\lambda_0^{D,\mathrm{ess}}(M)$, respectively. According to the next remark, Dirichlet extensions of Schrödinger operators are closely related to Schrödinger operators on non-complete manifolds without boundary.

Remark 2.1. If M is a Riemannian manifold with boundary, then any $f \in C_c^\infty(M)$ vanishing on ∂M, can be approximated in $H^1(M)$ with smooth functions, compactly supported in the interior of M. Therefore, if S is a Schrödinger operator on M, then the Dirichlet extension of S coincides with the Friedrichs extension of S viewed as an operator in the interior of M.

The next standard theorem provides essential self-adjointness of Schrödinger operators, in case the underlying manifold is complete.

Theorem 2.2 ([24, Chapter 8]). *If M is complete without boundary, then the operator defined in (2.1) is essentially self-adjoint. If M is complete with boundary, then the operator defined in (2.3) is essentially self-adjoint.*

2.1 Bottom of the spectrum

Let $S = \Delta + V$ be a Schrödinger operator on a possibly non-connected Riemannian manifold M. It is worth to point out that we do not require M to have non-empty boundary, which yields that the following results also hold for manifolds without boundary (and most of them are already known in this case). If M has non-empty boundary, we denote by ν the inward pointing normal to ∂M. The aim of this section is to establish some convenient expressions for the bottom of the spectrum, and derive some straightforward applications to Riemannian coverings.

Proposition 2.3. *Any compactly supported smooth function belongs to $H_V(M)$. Moreover, any compactly supported Lipschitz function is in $H_V(M)$.*

Proof: If M has empty boundary, then any compactly supported Lipschitz function f belongs to $H_0^1(M)$. Since V is smooth, it is easy to see that any such f also belongs to $H_V(M)$. Therefore, it remains to prove the proposition for manifolds with non-empty boundary.

Let $f \in C_c^\infty(M)$. Then there exists a compact $K \subset \partial M$ and $\delta > 0$, such that the map $\Phi \colon K \times [0,\delta) \to M$, defined by $\Phi(x,t) := \exp_x(t\nu)$, is a diffeomorphism onto its image W_δ, and $\operatorname{supp} f \cap W_\delta \subset W_\delta^\circ$. For $0 < \delta_0 < \delta$, consider the Lipschitz function f_{δ_0}, which is equal to f outside W_{δ_0}, and $f_{\delta_0}(\Phi(x,t)) = f(\Phi(x,\delta_0))$ in W_{δ_0}. Let K_1 be a compact neighborhood of $\Phi(K \times \{\delta_0\})$ and K_2 a compact neighborhood of K_1, that does not intersect ∂M. Consider $\chi \in C_c^\infty(M)$, with $\chi = 1$ in K_1 and $\operatorname{supp}\chi \subset K_2$. Since χf_{δ_0} is Lipschitz and compactly supported in the interior of M, it follows that $\chi f_{\delta_0} \in H_V(M)$. Moreover, $(1-\chi) f_{\delta_0} \in C_c^\infty(M)$ and $\nu(f) = 0$ on ∂M. Therefore, $(1-\chi) f_{\delta_0} \in H_V(M)$, which yields that $f_{\delta_0} \in H_V(M)$. It is clear that $f_{\delta_0} \to f$ in $H_V(M)$, as $\delta_0 \to 0$, and in particular, $f \in H_V(M)$.

Let f be a compactly supported Lipschitz function on M. Consider a Riemannian manifold N of the same dimension, without boundary, containing M (for instance, glue cylinders along ∂M). Extend f to a compactly supported Lipschitz function f' in N and let K be a smoothly bounded, compact neighborhood of $\operatorname{supp} f'$. Then there exists $(g_n)_{n\in\mathbb{N}} \subset C_c^\infty(N)$, with $\operatorname{supp} g_n \subset K$ and $g_n \to f'$ in $H_0^1(K)$. Then $h_n := g_n|_M \in C_c^\infty(M)$ and from the first statement, it follows that $h_n \in H_V(M)$. Evidently, we have that $h_n \to f$ in $H_V(M)$, and in particular, $f \in H_V(M)$. ∎

For $f \in \operatorname{Lip}_c(M) \smallsetminus \{0\}$, the *Rayleigh quotient* of f with respect to S, is defined as
$$\mathcal{R}_S(f) := \frac{\int_M (\|\operatorname{grad} f\|^2 + V f^2)}{\int_M f^2}.$$
It is worth to point out that this definition does not coincide completely with the notion of Rayleigh quotient introduced in Section 1.1. In virtue of the next proposition, these quantities indeed behave like Rayleigh quotients in Proposition 1.3.

Proposition 2.4. *The bottom of the spectrum of S^N is given by*
$$\lambda_0^N(S) = \inf_{f \in C_c^\infty(M) \smallsetminus \{0\}} \mathcal{R}_S(f) = \inf_{f \in \operatorname{Lip}_c(M) \smallsetminus \{0\}} \mathcal{R}_S(f).$$

Proof: It is clear that for any non-zero $f \in \operatorname{Lip}_c(M)$, we have
$$\mathcal{R}_S(f) = \inf_M V - 1 + \frac{\|f\|^2_{H_V(M)}}{\|f\|^2_{L^2(M)}},$$

and the asserted equalities follow from Proposition 1.5. ∎

Proposition 2.5. *Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_2 possibly non-connected. Let S_1 be a Schrödinger operator on M_1 and consider its lift S_2 on M_2. Then $\lambda_0^N(S_1) \leq \lambda_0^N(S_2)$.*

Proof: Let $f \in C_c^\infty(M_2) \smallsetminus \{0\}$ and consider its pushdown

$$g(z) := \Big(\sum_{y \in p^{-1}(z)} f(y)^2 \Big)^{1/2}$$

on M_1. Then $g \in \mathrm{Lip}_c(M_1)$, $\|g\|_{L^2(M_1)} = \|f\|_{L^2(M_2)}$ and $\mathcal{R}_{S_1}(g) \leq \mathcal{R}_{S_2}(f)$ (cf. [3, Section 4]). The statement follows from Proposition 2.4. ∎

Remark 2.6. It is worth to point out that the manifolds in this proposition are not required to have non-empty boundary. In particular, the analogous inequality holds for the bottoms of the spectra of Schrödinger operators on manifolds without boundary. Moreover, since the manifolds are not required to be complete, from Remark 2.1, it follows that the analogous inequality holds for the bottoms of the Dirichlet spectra of Schrödinger operators on manifolds with boundary.

We end this subsection with the next proposition, which characterizes the bottom of the spectrum of a Schrödinger operator as the maximum of its positive spectrum, and may be found for instance, in [10, Theorem 7], [14, Theorem 1] and [22, Theorem 2.1].

Proposition 2.7. *Let S be a Schrödinger operator on a complete Riemannian manifold M without boundary. Then $\lambda_0(S)$ is the maximum of all $\lambda \in \mathbb{R}$, such that there exists a positive $\varphi \in C^\infty(M)$, with $S\varphi = \lambda\varphi$.*

In particular, there exists a positive $\varphi \in C^\infty(M)$, with $S\varphi = \lambda_0(S)\varphi$. It is worth to point out that the positive functions involved in this proposition are not required to be square-integrable.

2.2 Eigenfunctions corresponding to the bottom

In this section we study properties of eigenfunctions corresponding to the bottom of the spectrum and minimizing sequences for the Rayleigh quotient of Schrödinger operators on connected Riemannian manifolds.

Proposition 2.8. *Let $S = \Delta + V$ be a Schrödinger operator on a Riemannian manifold M, and consider $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_c(M)$, with $\|f_n\|_{L^2(M)} = 1$ and $\mathcal{R}_S(f_n) \to \lambda_0^N(S)$. If $\lambda_0^N(S) \notin \sigma_{\mathrm{ess}}^N(S)$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$, such that $f_{n_k} \to \varphi$ in $L^2(M)$, for some $\lambda_0^N(S)$-eigenfunction φ of S^N.*

Proof: From Proposition 2.3, there exists $(f'_n)_{n\in\mathbb{N}} \subset C^\infty_c(M) \cap \mathcal{D}(S^N)$, such that $\|f'_n\|_{L^2(M)} = 1$ and $\|f_n - f'_n\|_{H_V(M)} \leq 1/n$, for any $n \in \mathbb{N}$. It is easy to see that $\mathcal{R}_S(f'_n) \to \lambda_0^N(S)$ and it suffices to prove the asserted statement for $(f'_n)_{n\in\mathbb{N}}$.

Since $\lambda_0^N(S)$ is not in the essential spectrum, it is an isolated eigenvalue of finite multiplicity. Let E be the eigenspace corresponding to $\lambda_0^N(S)$, and g_n be the projection of f'_n (with respect to the $L^2(M)$-inner product) on E, $n \in \mathbb{N}$. Since E is finite dimensional, after passing to a subsequence, we may assume that $g_n \to \varphi$ in $L^2(M)$, for some $\varphi \in E$. Consider $h_n := f'_n - g_n \in \mathcal{D}(S^N)$. Since h_n is perpendicular to E, from the Spectral Theorem (cf. for instance [24, Chapter 8]), it follows that there exists $c_0 > 0$, such that

$$\begin{aligned} \|h_n\|^2_{H_V(M)} - (1 - \inf_M V)\|h_n\|^2_{L^2(M)} &= \langle S^N h_n, h_n\rangle_{L^2(M)} \\ &\geq (\lambda_0^N(S) + c_0)\|h_n\|^2_{L^2(M)}, \end{aligned} \tag{2.4}$$

for any $n \in \mathbb{N}$. It is clear that

$$\begin{aligned} \langle h_n, g_n\rangle_{H_V(M)} &= \langle h_n, S^N g_n\rangle_{L^2(M)} + (1 - \inf_M V)\langle h_n, g_n\rangle_{L^2(M)} \\ &= (\lambda_0^N(S) + 1 - \inf_M V)\langle h_n, g_n\rangle_{L^2(M)} = 0. \end{aligned}$$

Let $\varepsilon > 0$. Then, for n sufficiently large, we have $\mathcal{R}_S(f'_n) \leq \lambda_0^N(S) + \varepsilon$, and thus

$$\begin{aligned} \|h_n\|^2_{H_V(M)} - (1 - \inf_M V)\|h_n\|^2_{L^2(M)} &= (\|f'_n\|^2_{H_V(M)} - (1 - \inf_M V)\|f'_n\|^2_{L^2(M)}) \\ &- (\|g_n\|^2_{H_V(M)} - (1 - \inf_M V)\|g_n\|^2_{L^2(M)}) \\ &\leq (\lambda_0^N(S) + \varepsilon)\|f'_n\|^2_{L^2(M)} - \lambda_0^N(S)\|g_n\|^2_{L^2(M)} \\ &= \varepsilon + \lambda_0^N(S)\|h_n\|^2_{L^2(M)}. \end{aligned}$$

From (2.4), this yields that $h_n \to 0$ in $L^2(M)$. Therefore, $f'_n \to \varphi$ in $L^2(M)$. ∎

Lemma 2.9. *Let S be a Schrödinger operator on a (connected) Riemannian manifold M and let $\varphi \in C^\infty(M) \smallsetminus \{0\}$ be a non-negative function satisfying $S\varphi = \lambda\varphi$, for some $\lambda \in \mathbb{R}$. Then φ is positive in the interior of M. If, in addition, M has non-empty boundary, and $\nu(\varphi) = 0$ on ∂M, then φ is positive on ∂M.*

Proof: Assume that there exists a point x in the interior of M, such that $\varphi(x) = 0$. Let $\delta > 0$, such that $\exp_x \colon B(0, 2\delta) \subset T_x M \to M$ is a diffeomorphism onto its image. Then $B(x,\delta)$ may be considered as a geodesic ball of radius δ in a complete Riemannian manifold without boundary. In $B(x,\delta)$, for any $\varepsilon > 0$, we have

$$|\Delta(\varphi + \varepsilon)| \leq (\varphi + \varepsilon) \sup_{B(x,\delta)} |\lambda - V|$$

and

$$\|\operatorname{grad}\Delta(\varphi+\varepsilon)\| \leq \|\operatorname{grad}(\varphi+\varepsilon)\| \sup_{B(x,\delta)} |\lambda - V| + (\varphi+\varepsilon) \sup_{B(x,\delta)} \|\operatorname{grad} V\|.$$

From [10, Theorem 6], it follows that there exists $c > 0$, independent from ε, such that

$$\sup_{B(x,\delta/2)} (\varphi+\varepsilon) \leq c \inf_{B(x,\delta/2)} (\varphi+\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this yields that if $\varphi(x) = 0$, then $\varphi = 0$ in $B(x,\delta/2)$. In particular, the set $\{x \in M^\circ : \varphi(x) = 0\}$ is open and closed. Since M is connected and φ is non-zero, it follows that φ is positive in M°.

Assume that M has non-empty boundary and $\nu(\varphi) = 0$ on ∂M. Assume that there exists $x \in \partial M$, such that $\varphi(x) = 0$. Since $\nu(\varphi) = 0$ on ∂M and $\varphi|_{\partial M}$ attains a minimum at x, it follows that $\operatorname{grad}\varphi(x) = 0$. Consider a coordinate system $\Phi\colon U := B(0,r) \cap \mathbb{H}^m \to M$, with $\Phi(0) = x$, where m is the dimension of M and $\mathbb{H}^m$ is the upper half-space of dimension m. Consider $c_0 \in \mathbb{R}$, with $c_0 \geq -\inf_M V$ and $c_0 \geq -\lambda$. Then $\phi := \varphi \circ \Phi$ is non-negative, smooth and satisfies

$$L\phi := -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\Big(g^{ij}\sqrt{\det g}\frac{\partial\phi}{\partial x_j}\Big) + (V+c_0)\phi = (\lambda+c_0)\phi \geq 0.$$

Since $V + c_0 \geq 0$, $\phi(0) = 0 < \phi(y)$ for all $y \in U^\circ$, and U satisfies the interior ball condition at the origin, from Hopf's Lemma (cf. for instance [13, p. 330]), it follows that

$$\frac{\partial\phi}{\partial x_m}(0) \neq 0,$$

which is a contradiction, since $\operatorname{grad}\phi(0) = 0$. Therefore, φ is positive on ∂M. ∎

Proposition 2.10. *Let M be a (connected) Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on M. If $\varphi \in \mathcal{D}(S^N) \smallsetminus \{0\}$ is a $\lambda_0^N(S)$-eigenfunction of S^N, then φ is smooth and nowhere vanishing. Moreover, if M has non-empty boundary, then $\nu(\varphi) = 0$ on ∂M.*

Proof: Since $\varphi \in \mathcal{D}(S^N)$, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)$, such that $f_n \to \varphi$ in $H_V(M)$. Clearly, $|f_n|$ is Lipschitz and compactly supported. From Proposition 2.3, it follows that $|f_n| \in H_V(M)$. From Rademacher's Theorem, $|f_n|$ is almost everywhere differentiable. Therefore, we have $\|\operatorname{grad}|f_n|\| = \|\operatorname{grad} f_n\|$ almost everywhere, and in particular, $\mathcal{R}_S(|f_n|) = \mathcal{R}_S(f_n)$. Since $(|f_n|)_{n\in\mathbb{N}}$ is bounded in $H_V(M)$, it has a weakly convergent subsequence in $H_V(M)$. Since $|f_n| \to |\varphi|$ in $L^2(M)$, it follows that $|\varphi| \in H_V(M)$, and after passing to a subsequence, we have that $|f_n| \rightharpoonup |\varphi|$ in $H_V(M)$. Hence, $\mathcal{R}_S(|\varphi|) = \lambda_0^N(S)$.

In particular, for any $f \in C_c^\infty(M)$, the function $t \mapsto \mathcal{R}_S(|\varphi| + tf)$, with $|t| < \varepsilon$, is differentiable and attains minimum for $t = 0$. This yields that

$$\int_M (\langle \operatorname{grad} |\varphi|, \operatorname{grad} f \rangle + V|\varphi| f) = \lambda_0^N(S) \int_M |\varphi| f, \tag{2.5}$$

for any $f \in C_c^\infty(M)$. From Elliptic Regularity Theory, it follows that $|\varphi| \in C^\infty(M^\circ)$ and $S|\varphi| = \lambda_0^N(S)|\varphi|$ in M°. From Lemma 2.9, $|\varphi|$ is nowhere vanishing in the interior of M, and so is φ. If M has empty boundary, this completes the proof.

If M has non-empty boundary, then without loss of generality, we may assume that φ is positive in the interior of M. Since $\varphi \in \mathcal{D}(S^N)$ and $S^N\varphi = \lambda_0^N(S)\varphi$, from Elliptic Regularity Theory, it follows that $\varphi \in C^\infty(M)$. Moreover, from (2.5) we have that

$$\int_{\partial M} \nu(\varphi) f = \int_M f S\varphi - \int_M (\langle \operatorname{grad} \varphi, \operatorname{grad} f \rangle + V\varphi f) = 0,$$

for any $f \in C_c^\infty(M)$. Therefore, $\nu(\varphi) = 0$ on ∂M, and from Lemma 2.9, it follows that φ is positive on ∂M. ∎

Proposition 2.11. *Let S be a Schrödinger operator on a (connected) Riemannian manifold M, with $\lambda_0^N(S) \notin \sigma_{\mathrm{ess}}^N(S)$. Then for any compact $K \subset M$ of positive measure, we have*

$$\inf_f \mathcal{R}_S(f) > \lambda_0^N(S),$$

where the infimum is taken over all non-zero $f \in \operatorname{Lip}_c(M)$, with $\operatorname{supp} f \cap K = \emptyset$.

Proof: Assume to the contrary that there exists a compact $K \subset M$ of positive measure, such that for any $\varepsilon > 0$, there exists a non-zero $f \in \operatorname{Lip}_c(M)$, with $\mathcal{R}_S(f) < \lambda_0^N(S) + \varepsilon$ and $\operatorname{supp} f \cap K = \emptyset$. Then, there exists $(f_n)_{n\in\mathbb{N}} \subset \operatorname{Lip}_c(M)$, with $\|f_n\|_{L^2(M)} = 1$, $\operatorname{supp} f_n \cap K = \emptyset$ and $\mathcal{R}_S(f_n) \to \lambda_0^N(S)$. From Proposition 2.8, after passing to a subsequence, we have that $f_n \to \varphi$ in $L^2(M)$, for some $\lambda_0^N(S)$-eigenfunction φ of S^N. Since $\|\varphi\|_{L^2(M)} = 1$, from Proposition 2.10, it follows that φ is nowhere vanishing in M. This is a contradiction, since

$$\|\varphi - f_n\|_{L^2(M)}^2 \geq \int_K \varphi^2 > 0,$$

while $f_n \to \varphi$ in $L^2(M)$. This proves the asserted claim. ∎

2.3 Essential spectrum of Schrödinger operators

An important tool in the study of the essential spectrum of Schrödinger operators is the following theorem, which is known as the Decomposition Principle.

Theorem 2.12 ([11]). *Let S be a Schrödinger operator on a complete Riemannian manifold M without boundary. Then for any compact subset K of M, we have*

$$\sigma_{\text{ess}}(S) = \sigma_{\text{ess}}(S, M \smallsetminus K),$$

where $\sigma_{\text{ess}}(S, M \smallsetminus K)$ stands for the essential spectrum of the Friedrichs extension of S viewed as an operator on $M \smallsetminus K$.

There are various generalizations of this theorem. For instance, in [4], it is extended to a quite general class of differential operators on Riemannian manifolds with possibly non-empty boundary.

The following well-known characterization for points of the essential spectrum of a Schrödinger operator is an immediate consequence of Theorem 2.12, and may be found, for instance, in [1] .

Proposition 2.13. *Let S be a Schrödinger operator on a complete Riemannian manifold M without boundary, and consider $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{\text{ess}}(S)$ if and only if there exists $(f_n)_{n\in\mathbb{N}} \subset C^\infty_c(M)$, such that $\|f_n\|_{L^2(M)} = 1$, $(S - \lambda)f_n \to 0$ in $L^2(M)$, and for every compact $K \subset M$, there exists $n_0 \in \mathbb{N}$, such that* $\operatorname{supp} f_n \cap K = \emptyset$, *for all $n \geq n_0$.*

The following standard expression for the bottom of the essential spectrum (which may be found for instance, in [6, Proposition 3.2]) follows from Theorem 2.12, Propositions 1.3 and 1.4. Recall that this quantity is infinite when the spectrum is discrete.

Proposition 2.14 ([6, Proposition 3.2]). *Let S be a Schrödinger operator on a complete Riemannian manifold M without boundary. Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of M consisting of compact subsets of M. Then the bottom of the essential spectrum of S is given by*

$$\lambda_0^{\text{ess}}(S) = \lim_n \lambda_0(S, M \smallsetminus K_n),$$

where $\lambda_0(S, M \smallsetminus K_n)$ stands for the bottom of the spectrum of S viewed as an operator on $M \smallsetminus K_n$.

2.4 Renormalized Schrödinger operators

In this section we discuss the notion of renormalized Schrödinger operators, which was introduced for the Laplacian on complete manifolds without boundary in [8].

Let M be a possibly non-connected Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on M. Let $\varphi \in C^\infty(M)$ be a positive function, satisfying $S\varphi = \lambda\varphi$, for some $\lambda \in \mathbb{R}$. If M has non-empty boundary, assume that $\nu(\varphi) = 0$ on ∂M, where ν is the inward pointing normal to ∂M. Consider the space

$$L^2_\varphi(M) := \{[v] : \varphi v \in L^2(M)\},$$

where two measurable functions are equivalent if they are almost everywhere equal, endowed with the inner product $\langle v_1, v_2 \rangle_{L^2_\varphi(M)} := \int_M v_1 v_2 \varphi^2$. Then the map $\mu_\varphi \colon L^2_\varphi(M) \to L^2(M)$, defined by $\mu_\varphi v := \varphi v$, is an isometric isomorphism. In particular, $L^2_\varphi(M)$ is a separable Hilbert space. The *renormalization* S_φ of S with respect to φ is defined by

$$S_\varphi v := \mu_\varphi^{-1}(S^N - \lambda)(\mu_\varphi v), \text{ for all } v \in \mathcal{D}(S_\varphi) := \mu_\varphi^{-1}(\mathcal{D}(S^N)).$$

It is clear that the operator $S_\varphi \colon \mathcal{D}(S_\varphi) \subset L^2_\varphi(M) \to L^2_\varphi(M)$ is self-adjoint and $\sigma(S_\varphi) = \sigma^N(S) - \lambda$. For a non-zero $f \in \mathrm{Lip}_c(M)$, the *Rayleigh quotient* of f with respect to S_φ is defined as

$$\mathcal{R}_{S_\varphi}(f) := \frac{\int_M \|\operatorname{grad} f\|^2 \varphi^2}{\int_M f^2 \varphi^2}.$$

Again, this definition does not coincide completely with the notion of Rayleigh quotient introduced in Section 1.1. In virtue of the next propositions, these quantities indeed behave like Rayleigh quotients in Proposition 1.3.

Proposition 2.15. *In the above situation, if M has non-empty boundary, then the bottom of the spectrum of S_φ is given by*

$$\lambda_0^N(S) - \lambda = \lambda_0(S_\varphi) = \inf_f \mathcal{R}_{S_\varphi}(f),$$

where the infimum is taken over all non-zero $f \in C^\infty_c(M)$, with $\nu(f) = 0$ on ∂M.

Proof: Let $f \in C^\infty_c(M) \smallsetminus \{0\}$, with $\nu(f) = 0$ on ∂M. Since φ is smooth and $\nu(\varphi) = 0$ on ∂M, it follows that $f \in \mathcal{D}(S_\varphi)$. It is easy to see that

$$S_\varphi f = \Delta f - \frac{2}{\varphi}\langle \operatorname{grad} \varphi, \operatorname{grad} f \rangle.$$

Hence, we have

$$\begin{aligned}
\langle S_\varphi f, f \rangle_{L^2_\varphi(M)} &= \int_M (\varphi^2 f \Delta f - 2 f \varphi \langle \operatorname{grad} f, \operatorname{grad} \varphi \rangle) \\
&= \int_M (\langle \operatorname{grad}(\varphi^2 f), \operatorname{grad} f \rangle - 2 f \varphi \langle \operatorname{grad} f, \operatorname{grad} \varphi \rangle) + \int_{\partial M} \varphi^2 f \nu(f) \\
&= \int_M \|\operatorname{grad} f\|^2 \varphi^2,
\end{aligned}$$

where we used that $\nu(f) = 0$ on ∂M. In particular, we have that

$$\mathcal{R}_{S_\varphi}(f) = \frac{\langle S_\varphi f, f\rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}}.$$

From Proposition 1.3, it follows that $\mathcal{R}_{S_\varphi}(f) \geq \lambda_0(S_\varphi)$. From (1.2), it follows that there exists $(g_n)_{n\in\mathbb{N}} \subset C^\infty_c(M) \smallsetminus \{0\}$, with $\nu(g_n) = 0$ on ∂M and $\mathcal{R}_S(g_n) \to \lambda_0^N(S)$. Consider $f_n := \mu_\varphi^{-1} g_n$. It is clear that $f_n \in C^\infty_c(M)$, $\nu(f_n) = 0$ on ∂M, and $\mathcal{R}_{S_\varphi}(f_n) \to \lambda_0(S_\varphi)$. This proves the asserted equality. ∎

Proposition 2.16. *In the above situation, if M has empty boundary, then the bottom of the spectrum of S_φ is given by*

$$\lambda_0(S) - \lambda = \lambda_0(S_\varphi) = \inf_{f\in C^\infty_c(M)\smallsetminus\{0\}} \mathcal{R}_{S_\varphi}(f) = \inf_{f\in \mathrm{Lip}_c(M)\smallsetminus\{0\}} \mathcal{R}_{S_\varphi}(f).$$

Proof: From the definition of S_φ, it is easy to see that for a non-zero $f \in C^\infty_c(M)$, we have

$$\mathcal{R}_{S_\varphi}(f) = \frac{\langle S_\varphi f, f\rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}}.$$

As in the proof of the previous proposition, from Propositions 1.3 and 2.4, we obtain the middle asserted equality.

Let $f \in \mathrm{Lip}_c(M) \smallsetminus \{0\}$ and let K be a smoothly bounded compact neighborhood of $\operatorname{supp} f$. Since $f \in H^1_0(K)$, it follows that there exists $(g_n)_{n\in\mathbb{N}} \subset C^\infty(K)$, with $\operatorname{supp} g_n \subset K^\circ$, such that $g_n \to f$ in $H^1(M)$. Since φ is smooth and K is compact, it follows that $\mathcal{R}_{S_\varphi}(g_n) \to \mathcal{R}_{S_\varphi}(f)$, which proves the last asserted equality. ∎

CHAPTER 3

Spectrum under Riemannian coverings

In this chapter we investigate the behavior of the spectrum under assumptions on the Riemannian covering. Throughout this chapter, we work in the following context. Let $p\colon M_2 \to M_1$ be a Riemannian covering, $E_1 \to M_1$ a Riemannian or Hermitian vector bundle of rank κ, endowed with a (not necessarily metric) connection ∇, and $D_1\colon \Gamma(E_1) \to \Gamma(E_1)$ a differential operator on E_1. Let $E_2 \to M_2$ be the pullback bundle, endowed with the corresponding metric and connection ∇, and $D_2\colon \Gamma(E_2) \to \Gamma(E_2)$ the lift of D_1. If M_1 has empty boundary, we consider the space of compactly supported smooth sections of E_i as the domain of D_i, that is, $\mathcal{D}(D_i) := \Gamma_c(E_i)$, $i = 1,2$. If M_1 has non-empty boundary, the domain of D_i is the space

$$\mathcal{D}(D_i) := \{\eta \in \Gamma_c(E_i) : a_i\nabla_{\nu_i}\eta + b_i\eta = 0 \text{ on } \partial M_i\},$$

where ν_i is the inward pointing normal to ∂M_i, $i = 1,2$, a_1, b_1 are real or complex valued functions (depending on whether the bundles are Riemannian or Hermitian) on ∂M_1, and $a_2 = a_1 \circ p$, $b_2 = b_1 \circ p$. It is worth to point out that we do not impose any assumptions on a_1 and b_1. If D_1 is of order one, then we require $a_1 = 0$. When we refer to closability, symmetry or essential self-adjointness of D_i, we consider the operator

$$D_i\colon \mathcal{D}(D_i) \subset L^2(E_i) \to L^2(E_i),$$

$i = 1,2$. From Lemma 1.18, the operator D_i is closable and we denote by $\overline{D}_i$ its closure, $i = 1,2$. The main goal of this chapter is to prove the following theorems.

Theorem 3.1. *Assume that D_1 is essentially self-adjoint and let D_2' be a self-adjoint extension of D_2. If the covering is infinite sheeted and amenable, then $\sigma(\overline{D}_1) \subset \sigma_{\mathrm{ess}}(D_2')$.*

Theorem 3.2. *Assume that D_i is symmetric and bounded from below, and denote by $D_i^{(F)}$ its Friedrichs extension, $i = 1,2$. If the covering is infinite sheeted and amenable, then $\lambda_0^{\mathrm{ess}}(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$.*

3.1 Spectrum under amenable coverings

In this section, we study the behavior of the spectrum under infinite sheeted amenable coverings and in particular, establish the main results of this chapter.

3.1.1 Partition of unity

In this subsection, we construct a special partition of unity, which is used in the sequel to obtain cut-off functions on M_2.

Consider the universal coverings $p_i \colon \tilde{M} \to M_i$ and denote by Γ_i the deck transformations group of p_i, $i = 1,2$. If M_1 has empty boundary, consider a Riemannian metric h, conformal to the original metric g, such that (M_1, h) is complete. If M_1 has non-empty boundary, consider a Riemannian manifold N_1 containing M_1, as in Proposition 1.15, and a Riemannian metric h, conformal to the original metric g, such that (N_1, h) is complete. From now on, geodesics are considered with respect to h and its lifts. We denote by $\operatorname{grad} f$ and $\operatorname{grad}_{\mathrm{h}} f$ the gradient of a function f with respect to g and h (or their lifts), respectively. If M_1 has empty boundary, distances are considered with respect to h or its lifts. In this case, we denote the open (respectively, closed) ball of radius r around a point z by $B(z,r)$ (respectively, $C(z,r)$). If M_1 has non-empty boundary, the distance between two points is considered in (N_1, h) or its corresponding covering space. In this case, $B(z,r)$ and $C(z,r)$ stand for the corresponding balls in M_1, M_2 or $\tilde{M}$.

Fix $x \in M_1^\circ$, $u \in p_1^{-1}(x)$ and $r > 0$ large enough, so that $B(u,r) \cap \partial\tilde{M} \neq \emptyset$, if M_1 has non-empty boundary.

Lemma 3.3. *There exists a non-negative $\psi_u \in C_c^\infty(\tilde{M})$, such that* $\operatorname{supp} \psi_u \subset C(u, r+1)$ *and $\psi_u = 1$ in $C(u, r+1/2)$. Moreover, if M_1 has non-empty boundary, ψ_u can be chosen such that* $\operatorname{grad} \psi_u$ *is tangential to $\partial\tilde{M}$.*

Proof: It is clear that there exists a non-negative $\psi_u' \in C_c^\infty(\tilde{M})$ with $\psi_u' = 1$ in $C(u, r+1/2)$ and $\operatorname{supp} \psi_u' \subset C(u, r+1)$. If M_1 has empty boundary, this is the desired function. Otherwise, let $K := \partial\tilde{M} \cap C(u, r+2)$ and denote by ν the inward pointing normal to $\partial\tilde{M}$ with respect to the lift of h. Since K is compact, there exists $\varepsilon > 0$, with $\varepsilon < 1/8$, such that the map $\Phi \colon K \times [0, 2\varepsilon) \to \tilde{M}$, defined by

$$\Phi(x,t) := \exp_x(t\nu)$$

is a diffeomorphism onto its image K_ε. Let $K_1 := \partial\tilde{M} \cap C(u, r + 1/2 + 2\varepsilon)$ and $K_2 := \partial\tilde{M} \cap C(u, r + 1 - 2\varepsilon)$. Clearly, there exists a non-negative $\tau \in C_c^\infty(\partial\tilde{M})$, with $\operatorname{supp} \tau \subset K_2$ and $\tau = 1$ in K_1. Extend it to τ' in K_ε by $\tau'(\Phi(x,t)) := \tau(x)$, for all $(x,t) \in K \times [0, 2\varepsilon)$. Consider a smooth function $f \colon \mathbb{R} \to \mathbb{R}$, with $0 \leq f \leq 1$,

$f(x) = 1$ for $x \leq \varepsilon$, and $f(x) = 0$ for $x \geq 3\varepsilon/2$, and the function h defined in K_ε by $h(\Phi(x,t)) = f(t)$, for all $(x,t) \in K \times [0, 2\varepsilon)$. Extend h by zero outside K_ε and set

$$\psi_u := h\tau' + (1-h)\psi_u'.$$

Since $h\tau'$ and ψ_u' are supported in $C(u, r+1)$, it follows that $\operatorname{supp}\psi_u \subset C(u, r+1)$. Since $\varepsilon < 1/8$, the points where h is not smooth are not in $C(u, r+1)$, which yields that $\psi_u \in C_c^\infty(\tilde{M})$. Since $\psi_u' = 1$ in $C(u, r+1/2)$ and $\tau' = 1$ in $C(u, r+1/2) \cap K_\varepsilon$, it follows that $\psi_u = 1$ in $C(u, r+1/2)$. In $\Phi(K \times [0, \varepsilon))$, which is a neighborhood of $\operatorname{supp}\psi_u \cap \partial\tilde{M}$, we have $\psi_u = \tau'$. In particular, $\operatorname{grad}_{\mathrm{h}}\psi_u$ is tangential to $\partial\tilde{M}$, and so is $\operatorname{grad}\psi_u$, since g and h are conformal. ∎

Let ψ_u be a function as in the above lemma. For $y \in p^{-1}(x)$, fix $u(y) \in p_2^{-1}(y)$ and $g(y) \in \Gamma_1$, such that $u(y) = g(y)u$. Consider the functions $\psi_{u(y)} := \psi_u \circ g(y)^{-1}$ in $\tilde{M}$ and ψ_y in M_2 defined by

$$\psi_y(z) := \sum_{w \in p_2^{-1}(z)} \psi_{u(y)}(w). \tag{3.1}$$

It is clear that $\psi_y \in C_c^\infty(M_2)$, $\operatorname{supp}\psi_y \subset C(y, r+1)$ and $\psi_y \geq 1$ in $C(y, r+1/2)$, for any $y \in p^{-1}(x)$. Moreover, if M_1 has non-empty boundary, then $\operatorname{grad}\psi_y$ is tangential to ∂M_2, for all $y \in p^{-1}(x)$. From Lemma 1.7, there exists $N(r+2) \in \mathbb{N}$, such that for any $z \in M_2$, the ball $B(z,1)$ intersects at most $N(r+2)$ of the supports of ψ_y, with $y \in p^{-1}(x)$. Therefore, $\sum_{y \in p^{-1}(x)} \psi_y$ is locally a finite sum and hence, well-defined and smooth.

If M_1 is compact, we choose r large enough, so that $\sum_{y \in p^{-1}(x)} \psi_y \geq 1$ in M_2. In this case, set $\psi_1 := 0$ in M_2. If M_1 is non-compact, consider $f_1 \in C_c^\infty(M_1)$ with $0 \leq f_1 \leq 1$, $f_1 = 1$ in $C(x,r)$, $\operatorname{supp} f_1 \subset B(x, r+1/2)$, and let ψ_1 be the lift of $1 - f_1$ on M_2. Then $\psi_1 \in C^\infty(M_2)$, $\psi_1 \geq 0$ in M_2 and $\psi_1 = 0$ in $C(y,r)$, for all $y \in p^{-1}(x)$. Evidently, $\psi_1 + \sum_{y \in p^{-1}(x)} \psi_y \geq 1$ in M_2.

Consider the smooth partition of unity consisting of the functions

$$\varphi_1 := \frac{\psi_1}{\psi_1 + \sum_{y' \in p^{-1}(x)} \psi_{y'}} \quad \text{and} \quad \varphi_y := \frac{\psi_y}{\psi_1 + \sum_{y' \in p^{-1}(x)} \psi_{y'}}, \tag{3.2}$$

with $y \in p^{-1}(x)$.

Remark 3.4. Evidently, $\operatorname{supp}\varphi_1 = \operatorname{supp}\psi_1$, $\operatorname{supp}\varphi_y = \operatorname{supp}\psi_y$, $\sum_{y' \in p^{-1}(x)} \varphi_{y'} = 1$ in $C(y,r)$ and $\varphi_y > 0$ in $C(y, r+1/2)$, for any $y \in p^{-1}(x)$. If M_1 has non-empty boundary, then for any $y, y' \in p^{-1}(x)$, we have that $\operatorname{grad}\psi_y$ is tangential to ∂M_2 and $\psi_1 = 0$ in $B(y', r)$. This yields that $\operatorname{grad}\varphi_y$ is tangential to ∂M_2 in $B(y', r)$, for all $y, y' \in p^{-1}(x)$.

Let $\eta \in \mathcal{D}(D_1)$ and $\theta \in \Gamma(E_2)$ be the lift of η. Fix $x \in M_1^\circ$, $u \in p_1^{-1}(x)$ and $r > 0$, such that $\operatorname{supp} \eta \subset B(x,r)$. If M_1 has non-empty boundary, we choose r large enough, so that $B(u,r) \cap \partial\tilde{M} \neq \emptyset$. Consider a partition of unity associated to x, u and r as in (3.2) and for a finite $P \subset p^{-1}(x)$, set $\chi := \sum_{y \in P} \varphi_y$.

Remark 3.5. Since P is finite, it follows that $\chi \in C_c^\infty(M_2)$ and $\chi\theta \in \Gamma_c(E_2)$. Since $\operatorname{supp} \eta \subset B(x,r)$, we have that $\operatorname{supp} \theta$ is contained in the union of the balls $B(y,r)$, with $y \in p^{-1}(x)$. Therefore, if M_1 has non-empty boundary, from Remark 3.4, $\chi\theta$ satisfies analogous boundary conditions to η, that is, $\chi\theta \in \mathcal{D}(D_2)$.

Proposition 3.6. *There exists a constant C, independent from P, such that the estimate $\|D_2(\chi\theta)(z)\| \leq C$ holds for any $z \in M_2$.*

Proof: Consider $\delta > 0$, such that for any $x' \in C(x, r+1)$, the ball $B(x', 2\delta)$ is evenly covered and contained in a coordinate neighborhood, and $E_1|_{B(x',2\delta)}$ is trivial. Let $x_1, \dots, x_k \in C(x, r+1)$, such that the balls $B(x_i, \delta)$, $1 \leq i \leq k$, cover $C(x, r+1)$. In any ball $B(x_i, 2\delta)$, D_1 has a local expression of the form (1.3), with A^α smooth. This yields that in $B(x_i, \delta)$, D_1 is expressed in the form (1.3), with A^α smooth and bounded. For any such ball, we fix a coordinate system (which can be extended to the corresponding ball of radius 2δ) and a trivialization. Since $C(x, r+1)$ is covered by finitely many such balls, it follows that there exists $C_1 > 0$, such that in any of these balls, we have $\|A^\alpha\| \leq C_1$, for all multi-indices α of absolute value less or equal to the order d of D_1.

Since η is smooth and compactly supported in $B(x,r)$, there exists $C_2 > 0$, such that in any of these balls, denoting by $(\eta^{(1)}, \dots, \eta^{(\kappa)})$ the local expression of η, we have that

$$\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}(\eta^{(1)}, \dots, \eta^{(\kappa)})\| \leq C_2,$$

for all multi-indices α of absolute value less or equal to d, that is, we have *uniform estimates up to order d for η (with respect to this system of trivializations)*. We lift these balls and the corresponding coordinate systems and trivializations to M_2 and $\tilde{M}$. Similarly, if $\psi_1 \neq 0$, we obtain uniform estimates up to order d for f_1, which yield uniform estimates up to order d for ψ_1 (with respect to the lifted system on M_2).

Since ψ_u is smooth and compactly supported in $C(u, r+1)$, which intersects finitely many balls of the lifted system on $\tilde{M}$, there exist uniform estimates up to order d for ψ_u. Since $\psi_{u(y)}$ is a composition of ψ_u with an element of Γ_1, we obtain the same uniform estimates up to order d for $\psi_{u(y)}$, for all $u(y)$. Recall the definition of ψ_y in (3.1). Consider a ball $B(z', \delta)$ of the lifted system on M_2, which intersects $\operatorname{supp} \psi_y$, and the corresponding coordinate system. Evidently, for any $w \in p_2^{-1}(z')$, the lifted system on $\tilde{M}$ contains the ball $B(w, \delta)$ and the corresponding coordinate system. From Lemma 1.8, there exists $\tilde{N}(r+1, \delta) \in \mathbb{N}$, independent from y and

z', such that at most $\tilde{N}(r+1,\delta)$ such balls intersect the support of $\psi_{u(y)}$. Since we have uniform estimates up to order d for $\psi_{u(y)}$, which are independent from $y \in p^{-1}(x)$, we obtain the same uniform estimates up to order d for ψ_y, for all $y \in p^{-1}(x)$. From Lemma 1.7, it follows that at most $N(r+1+\delta)$ of the supports of ψ_y, with $y \in p^{-1}(x)$, intersect the open ball $B(z,\delta)$, for any $z \in M_2$. This yields that there exist uniform estimates up to order d for $\psi_1 + \sum_{y \in p^{-1}(x)} \psi_y$.

Recall the definition of φ_y in (3.2). Since the denominator is greater or equal to 1 and we have uniform estimates (independent from y) up to order d for the numerator and the denominator, we obtain the same uniform estimates up to order d for φ_y, for all $y \in p^{-1}(x)$. From Lemma 1.7, at most $N(r+1+\delta)$ of the supports of φ_y, with $y \in p^{-1}(x)$, intersect the ball $B(z,\delta)$, for any $z \in M_2$. Therefore, we obtain uniform estimates up to order d for χ, which are independent from P

Clearly, for $z \in \operatorname{supp}(\chi\theta)$, we have that $z \in B(y,r)$, for some $y \in p^{-1}(x)$, and in particular, z is contained in a ball of the system. With respect to the corresponding coordinate system and trivialization, denoting by $(\theta^{(1)}, \dots, \theta^{(\kappa)})$ the local expression of θ, we have

$$\begin{aligned} \|D_2(\chi\theta)(z)\| &= \|\sum_{|\alpha|\le d} (A^\alpha \circ p)(z) \frac{\partial^{|\alpha|}}{\partial y^\alpha}(\chi(\theta^{(1)}, \dots, \theta^{(\kappa)}))(z)\| \\ &\le \sum_{|\alpha|\le d} C_1 \|\frac{\partial^{|\alpha|}}{\partial y^\alpha}(\chi(\theta^{(1)}, \dots, \theta^{(\kappa)}))(z)\| \\ &\le C_1 C_2 C_3 C(d,\kappa), \end{aligned}$$

where C_3 is the uniform bound up to order d for χ (which is independent from P) and $C(d,\kappa)$ is a constant depending only on d and κ. ∎

Corollary 3.7. *There exists a constant C', independent from P, such that for any $z \in M_2$, we have $|\langle D_2(\chi\theta)(z), (\chi\theta)(z)\rangle| \le C'$.*

Proof: Follows immediately from Proposition 3.6. ∎

3.1.2 Amenable coverings

In this subsection we continue to work in the setting of the previous subsection. In particular, we extend the covering $p\colon M_2 \to M_1$ to a Riemannian covering $p\colon N_2 \to N_1$ according to Proposition 1.15 (if needed) and consider conformal Riemannian metrics, such that the manifolds become complete. If M_1 has empty boundary, for $x \in M_1$ and $y \in p^{-1}(x)$, we denote by D_y the fundamental domain of $p\colon M_2 \to M_1$ centered at y, with respect to these conformal Riemannian metrics. If M_1 has non-empty boundary, we denote by D_y the part of the fundamental domain of $p\colon N_2 \to N_1$ that lies in M_2. Furthermore, volumes, integrals and L^2-norms are with respect to the original Riemannian metrics.

As in the previous subsection, consider the universal coverings $p_i\colon \tilde{M} \to M_i$, $i = 1, 2$, and fix $x \in M_1^\circ$ and $u \in p_1^{-1}(x)$. For $r > 0$, consider the finite set

$$G_r := \{g \in \pi_1(M_1) : \text{there exists a representative loop } \gamma_g \text{ of } g, \text{ with } \ell(\gamma_g) < r\},$$

where $\ell(\cdot)$ stands for the length of the curve. Denote by $\langle G_r \rangle$ the subgroup of $\pi_1(M_1)$ generated by G_r. We are interested in the right action of $\langle G_r \rangle$ on $p^{-1}(x)$. The next remark is a simple description of the orbits of this action.

Remark 3.8. Let $y \in p^{-1}(x)$ and $g \in G_r$. Then there exists a representative loop γ_g of g, of length less than r. In particular, we have

$$d(y, y \cdot g) \leq \ell(\gamma_g) < r.$$

Conversely, let $y_1, y_2 \in p^{-1}(x)$ with $d(y_1, y_2) < r$. Let $\gamma\colon [0,1] \to M_2$ be a smooth path from y_1 to y_2, of length less than r. Then $\gamma_g := p \circ \gamma$ is a representative of some $g \in \pi_1(M_1)$, which has length less than r. Evidently, $g \in G_r$ and $y_2 = y_1 \cdot g$.

Hence, two points $z_1, z_2 \in p^{-1}(x)$ are in the same orbit of the action of $\langle G_r \rangle$ on $p^{-1}(x)$ if and only if there exist $k \in \mathbb{N}$ and $y_1, \dots, y_k \in p^{-1}(x)$, such that $y_1 = z_1$, $y_k = z_2$ and $d(y_i, y_{i+1}) < r$, for $i = 1, \dots, k-1$.

Lemma 3.9. *If $p\colon M_2 \to M_1$ is infinite sheeted, then there exists $R > 0$, such that one of the following holds:*
(i) either for any $r \geq R$, the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has only infinite orbits,
(ii) or for any $r \geq R$, the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has infinitely many finite orbits.

Proof: Assume to the contrary that the statement does not hold. Then there exists $r_0 > 0$, such that the action of $\langle G_{r_0} \rangle$ on $p^{-1}(x)$ has only finitely many finite orbits $\mathcal{O}_1, \dots, \mathcal{O}_k$, for some $k \in \mathbb{N}$. Since p is infinite sheeted, there exists also an infinite orbit $\mathcal{O}$. Since the action of $\pi_1(M_1)$ on $p^{-1}(x)$ is transitive, for $y_i \in \mathcal{O}_i$, there exists $g_i \in \pi_1(M_1)$, such that $y_i \cdot g_i \in \mathcal{O}$, for $i = 1, \dots, k$. Then there exists $R > 0$, such that $G_{r_0} \cup \{g_1, \dots, g_k\} \subset G_R$ and the action of $\langle G_R \rangle$ on $p^{-1}(x)$ has only infinite orbits. It is clear that so does the action of $\langle G_r \rangle$ on $p^{-1}(x)$, for any $r \geq R$, which is a contradiction. ∎

Let $r > 0$ large enough, so that $B(u, r) \cap \partial\tilde{M} \neq \emptyset$, if M_1 has non-empty boundary. If p is infinite sheeted, we choose $r \geq R$, where R is the constant from Lemma 3.9. Consider a partition of unity consisting of the functions φ_1 and φ_y, with $y \in p^{-1}(x)$, associated to x, u and r as in (3.2). For a finite $P \subset p^{-1}(x)$, let

$\chi := \sum_{y \in P} \varphi_y$ and consider the sets

$$\begin{aligned} Q_+ &:= \{y \in p^{-1}(x) : \chi = 1 \text{ in } B(y,r)\} \\ Q_- &:= \{y \in p^{-1}(x) : 0 < \chi(z) < 1 \text{ for some } z \in B(y,r)\}, \\ Q &:= Q_+ \cup Q_- = \{y \in p^{-1}(x) : \chi(z) \neq 0 \text{ for some } z \in B(y,r)\}. \end{aligned} \tag{3.3}$$

Clearly, $\chi = 0$ in $B(y,r)$, for any $y \in p^{-1}(x) \smallsetminus Q$. Since χ is compactly supported, it follows that Q is finite. The proof of the following lemma is essentially presented in [3], but since we are in a different situation here, we repeat it.

Lemma 3.10. *If p is amenable, then for any $\varepsilon > 0$, there exists a non-empty, finite subset P of $p^{-1}(x)$, such that*

$$\frac{\#(Q_-)}{\#(Q_+)} < \varepsilon.$$

Proof: From Proposition 1.9, since p is amenable, for any $\delta > 0$, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that

$$\#(P \smallsetminus Pg) < \delta \#(P),$$

for all $g \in G_{2r+2}$. From Remark 3.4, we have that $\operatorname{supp} \varphi_{y_0} \subset C(y_0, r+1)$, $\varphi_{y_0} > 0$ in $B(y_0, r+1/2)$ and $\sum_{y \in p^{-1}(x)} \varphi_y = 1$ in $B(y_0, r)$, for any $y_0 \in p^{-1}(x)$. Clearly, P is contained in Q, which implies that $\#(P) \leq \#(Q)$.

For $y \in Q_-$, there exists $z \in B(y,r)$, such that $0 < \chi(z) < 1$. Therefore, there exist $y_1 \in P$ and $y_2 \in p^{-1}(x) \smallsetminus P$, such that $\varphi_{y_i}(z) > 0$, which yields that $d(y_i, z) < r+1$, $i = 1,2$. It follows that $d(y_1, y_2) < 2r+2$ and from Remark 3.8, there exists $g \in G_{2r+2}$, such that $y_1 = y_2 \cdot g$. In particular, $y_1 \in P \smallsetminus Pg$. Since $d(y, y_1) < 2r+1$, from Lemma 1.7, for a fixed y_1, there exist at most $N(2r+1)$ such y. Since $y_1 \in P \smallsetminus Pg$, for some $g \in G_{2r+2}$, there exist at most $\delta \#(P) \#(G_{2r+2})$ such y_1. Hence, it follows that

$$\#(Q_-) \leq \delta \#(P) \#(G_{2r+2}) N(2r+1) \leq \delta \#(Q) \#(G_{2r+2}) N(2r+1).$$

Since Q is the disjoint union of Q_+ and Q_-, for $\delta \#(G_{2r+2}) N(2r+1) < 1$, we have

$$\frac{\#(Q_-)}{\#(Q_+)} \leq \frac{\delta \#(G_{2r+2}) N(2r+1)}{1 - \delta \#(G_{2r+2}) N(2r+1)}.$$

This completes the proof, since $\delta > 0$ is arbitrarily small. ∎

Proposition 3.11. *If $p \colon M_2 \to M_1$ is infinite sheeted and amenable, then for any $\varepsilon > 0$ and $K \subset M_2$ compact, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that* $\operatorname{supp} \chi$ *does not intersect K and*

$$\frac{\#(Q_-)}{\#(Q_+)} < \varepsilon.$$

Proof: First assume that the second statement of Lemma 3.9 holds. Then the action of $\langle G_{2r+2} \rangle$ on $p^{-1}(x)$ has infinitely many finite orbits $\mathcal{O}_n$, with $n \in \mathbb{N}$. Clearly, for $P := \mathcal{O}_n$, we have that Q_- is empty. Indeed, if there exists $y_0 \in Q_-$, then there exist $z \in B(y_0, r)$, $y_1 \in P$ and $y_2 \in p^{-1}(x) \smallsetminus P$, such that $\varphi_{y_i}(z) > 0$, $i = 1, 2$. It follows that $d(z, y_i) < r + 1$, $i = 1, 2$, which yields that $d(y_1, y_2) < 2r + 2$. From Remark 3.8, there exists $g \in G_{2r+2}$, such that $y_2 = y_1 \cdot g$, which is a contradiction, since P is an orbit of the action of $\langle G_{2r+2} \rangle$ on $p^{-1}(x)$.

For a compact $K \subset M_2$, the set $P_K := p^{-1}(x) \cap B(K, r + 2)$ is finite and in particular, intersects only finitely many orbits $\mathcal{O}_n$. Let P be an orbit that does not intersect P_K. Since $\operatorname{supp} \varphi_y \subset C(y, r + 1)$, for any $y \in p^{-1}(x)$, it is clear that for such P, the support of χ does not intersect K.

Assume now that the first statement of Lemma 3.9 holds. Then the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has only infinite orbits. For a compact subset K of M_2, consider the finite set $P_K := p^{-1}(x) \cap B(K, r + 2)$. From Lemma 3.10, for any $\varepsilon > 0$, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that

$$\frac{\#(Q_-)}{\#(Q_+)} < \delta := \frac{\varepsilon}{1 + (1 + \varepsilon) N(2r + 1) \#(P_K)},$$

where $N(2r + 1)$ is the constant from Lemma 1.7.

Since the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has only infinite orbits, it follows that Q_- is non-empty. Indeed, since P is non-empty and this action has only infinite orbits, it is clear that there exists an infinite orbit $\mathcal{O}$ and $z_1 \in P \cap \mathcal{O}$. Since P is finite, there exists $z_2 \in \mathcal{O} \smallsetminus P$, and from Remark 3.8, there exist $k \in \mathbb{N}$ and $y_1, \dots, y_k \in p^{-1}(x)$, with $y_1 = z_1$, $y_k = z_2$ and $d(y_i, y_{i+1}) < r$, for $i = 1, \dots, k - 1$. Since $y_1 \in P$ and $y_k \notin P$, there exists $1 \leq j < k$, such that $y_j \in P$ and $y_{j+1} \notin P$. Since $d(y_j, y_{j+1}) < r$, it follows that $0 < \chi(y_{j+1}) < 1$ and in particular, $y_j \in Q_-$.

Evidently, Q_+ is contained in P. Since Q_- is non-empty, it is clear that

$$\frac{1}{\delta} \leq \#(Q_+) \leq \#(P),$$

which yields that $\#(P) > \#(P_K)$, from the choice of δ. In particular, the finite set $P' := P \smallsetminus P_K$ is non-empty. Consider the function χ' and the sets Q'_+, Q'_- and Q' corresponding to P' as in (3.3). Clearly, the support of χ' does not intersect K, since $\operatorname{supp} \varphi_y \subset C(y, r + 1)$, for any $y \in p^{-1}(x)$.

From Lemma 1.7, it follows that for any $y_0 \in p^{-1}(x)$, the support of φ_{y_0} intersects at most $N(2r + 1)$ open balls $B(y, r)$, with $y \in p^{-1}(x)$. Hence, we have that

$$\begin{aligned} \#(Q'_-) &\leq \#(Q_-) + N(2r + 1)\#(P_K), \\ \#(Q'_+) &\geq \#(Q_+) - N(2r + 1)\#(P_K). \end{aligned}$$

Therefore, we obtain

$$\frac{\#(Q'_-)}{\#(Q'_+)} \leq \frac{\#(Q_-) + N(2r+1)\#(P_K)}{\#(Q_+) - N(2r+1)\#(P_K)} < \varepsilon,$$

from the choice of δ. ∎

Remark 3.12. After endowing M_1 or N_1 with h (depending on whether M_1 has empty boundary or not) and the covering space with its lift, for any $y \in p^{-1}(x)$, we have that the restriction $p\colon D_y \to M_1$ is an isometry up to sets of measure zero. Therefore, for $f \in C_c(M_1)$, we have

$$\int_{D_y} (f \circ p) d\mathrm{Vol}_{\mathrm{h}_2} = \int_{M_1} f d\mathrm{Vol}_{\mathrm{h}_1}, \tag{3.4}$$

where $\mathrm{Vol}_{\mathrm{h}_i}$ (respectively, $\mathrm{Vol}_{\mathrm{g}_i}$) is the measure on M_i induced by h (respectively, g) or its lift, $i = 1, 2$. Since g and h are conformal, there exists a positive $\mathcal{V} \in C^\infty(M_1)$, such that

$$\frac{d\mathrm{Vol}_{\mathrm{h}_1}}{d\mathrm{Vol}_{\mathrm{g}_1}} = \mathcal{V} \text{ and } \frac{d\mathrm{Vol}_{\mathrm{h}_2}}{d\mathrm{Vol}_{\mathrm{g}_2}} = \mathcal{V} \circ p.$$

For simplicity of notation, we omit $d\mathrm{Vol}_{\mathrm{g}_i}$ in the integrals and the index of $\mathrm{Vol}_{\mathrm{g}_i}$. From (3.4), we have $\int_{D_y} f \circ p = \int_{M_1} f$, for any $f \in C_c(M_1)$ and $y \in p^{-1}(x)$. Similarly, for a compact $K \subset M_1$, we have $\mathrm{Vol}(K) = \mathrm{Vol}(p^{-1}(K) \cap D_y)$, for any $y \in p^{-1}(x)$.

Proposition 3.13. *Let $p\colon M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering. Let $\eta \in \mathcal{D}(D_1)$, with $\|\eta\|_{L^2(E_1)} = 1$, and $\lambda \in \mathbb{F}$. Then for any $\varepsilon > 0$ and $K \subset M_2$ compact, there exists $\zeta \in \mathcal{D}(D_2)$ with $\|\zeta\|_{L^2(E_2)} = 1$, such that $\operatorname{supp}\zeta \subset p^{-1}(\operatorname{supp}\eta)$, $\operatorname{supp}\zeta \cap K = \emptyset$ and $\|(D_2 - \lambda)\zeta\|_{L^2(E_2)} \leq \|(D_1 - \lambda)\eta\|_{L^2(E_1)} + \varepsilon$.*

Proof: Consider the universal covering $p_1\colon \tilde{M} \to M_1$ of M_1, fix $x \in M_1^\circ$, $u \in p_1^{-1}(x)$ and $r \geq R$ (where R is the constant from Lemma 3.9), such that $\operatorname{supp}\eta \subset B(x,r)$ and $B(u,r) \cap \partial\tilde{M} \neq \emptyset$, if M_1 has non-empty boundary. Consider a partition of unity consisting of the functions φ_1 and φ_y, with $y \in p^{-1}(x)$, associated to x, u and r as in (3.2), and let θ be the lift of η. From Remark 3.5, for any finite set $P' \subset p^{-1}(x)$ and $\chi' := \sum_{y \in P'} \varphi_y$, we have that $\chi'\theta \in \mathcal{D}(D_2)$. From Proposition 3.6, there exists $C > 0$, independent from P', such that $\|D_2(\chi'\theta)(z)\| \leq C$, for any $z \in M_2$. Hence, we obtain that

$$\max_{z \in M_2} \left\|(D_2 - \lambda)(\chi'\theta)(z)\right\| \leq C + |\lambda| \max_{w \in M_1} \|\eta(w)\| =: C_0.$$

From Proposition 3.11, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that the support of $\chi := \sum_{y \in P} \varphi_y$ does not intersect K and

$$\frac{\#(Q_-)}{\#(Q_+)} < \min\left\{ \frac{\varepsilon}{C_0^2 \mathrm{Vol}(\operatorname{supp}\eta)}, \varepsilon \right\},$$

where Q_+, Q_- and Q are the sets corresponding to P as in (3.3).

Since $\chi\theta$ is in the domain of D_2, so is the corresponding normalized section $\zeta := (1/\|\chi\theta\|_{L^2(E_2)})\chi\theta$. Evidently, $\|\zeta\|_{L^2(E_2)} = 1$ and $\operatorname{supp}\zeta \subset p^{-1}(\operatorname{supp}\eta)$. From Lemma 1.6, we have that $\operatorname{supp}\zeta \cap D_y \subset B(y,r)$, for any $y \in p^{-1}(x)$, which yields that $\operatorname{supp}\zeta$ is contained in the union of the fundamental domains D_y, with $y \in Q$. Clearly, we have

$$\|\chi\theta\|^2_{L^2(E_2)} \geq \sum_{y \in Q_+} \int_{D_y} \|\chi\theta\|^2 = \sum_{y \in Q_+} \int_{D_y} \|\theta\|^2 = \#(Q_+),$$

from the definition of Q_+ and Remark 3.12. Therefore, we obtain that

$$\begin{aligned} \int_{M_2} \|(D_2 - \lambda)\zeta\|^2 &\leq \frac{1}{\#(Q_+)} \sum_{y \in Q_+} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 \\ &+ \frac{1}{\#(Q_+)} \sum_{y \in Q_-} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2. \end{aligned}$$

For $y \in Q_+$, we have $\chi = 1$ in $B(y,r)$, which is a neighborhood of $\operatorname{supp}\theta \cap D_y$. This implies that

$$\begin{aligned} \frac{1}{\#(Q_+)} \sum_{y \in Q_+} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 &= \frac{1}{\#(Q_+)} \sum_{y \in Q_+} \int_{D_y} \|(D_2 - \lambda)\theta\|^2 \\ &= \int_{M_1} \|(D_1 - \lambda)\eta\|^2. \end{aligned}$$

Since $\|(D_2 - \lambda)(\chi\theta)(z)\| \leq C_0$, for any $z \in M_2$, it follows that

$$\begin{aligned} \frac{1}{\#(Q_+)} \sum_{y \in Q_-} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 &\leq \frac{C_0^2}{\#(Q_+)} \sum_{y \in Q_-} \mathrm{Vol}(\operatorname{supp}\theta \cap D_y) \\ &= \frac{\#(Q_-)}{\#(Q_+)} C_0^2 \mathrm{Vol}(\operatorname{supp}\eta) \leq \varepsilon. \end{aligned}$$

Hence, $\|(D_2 - \lambda)\zeta\|^2_{L^2(E_2)} \leq \|(D_1 - \lambda)\eta\|^2_{L^2(E_1)} + \varepsilon$. ∎

Proposition 3.14. *Let $p\colon M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering, and assume that the operators D_i are symmetric, $i = 1, 2$. Then for any section $\eta \in \mathcal{D}(D_1) \smallsetminus \{0\}$, $\varepsilon > 0$ and $K \subset M_2$ compact, there exists $\zeta \in \mathcal{D}(D_2) \smallsetminus \{0\}$, such that* $\operatorname{supp} \zeta \subset p^{-1}(\operatorname{supp} \eta)$, $\operatorname{supp} \zeta \cap K = \emptyset$ *and* $\mathcal{R}_{D_2}(\zeta) \leq \mathcal{R}_{D_1}(\eta) + \varepsilon$.

Proof: The proof is similar to the proof of Proposition 3.13, using Corollary 3.7 instead of Proposition 3.6. ∎

We are ready to prove the following more general version of Theorem 3.1.

Theorem 3.15. *Let D_2' be a closed extension of D_2. If the covering is infinite sheeted and amenable, then $\sigma_{\mathrm{ap}}(\overline{D}_1) \subset \sigma_W(D_2')$.*

Proof: Let $\lambda \in \sigma_{\mathrm{ap}}(\overline{D}_1)$. From Lemma 1.2, there exists $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D_1)$, such that $\|\eta_n\|_{L^2(E_1)} = 1$ and $(D_1 - \lambda)\eta_n \to 0$ in $L^2(E_1)$. Consider an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of M_2 consisting of compact subsets of M_2. From Proposition 3.13, for any $n \in \mathbb{N}$, there exists $\zeta_n \in \mathcal{D}(D_2)$, such that $\|\zeta_n\|_{L^2(E_2)} = 1$, $\operatorname{supp} \zeta_n \cap K_n = \emptyset$ and $\|(D_2 - \lambda)\zeta_n\|_{L^2(E_2)} \leq \|(D_1 - \lambda)\eta_n\|_{L^2(E_1)} + 1/n$. Therefore, $(D_2 - \lambda)\zeta_n \to 0$ in $L^2(E_2)$ and for any compact subset K of M_2, there exists $n_0 \in \mathbb{N}$, such that $\operatorname{supp} \zeta_n \cap K = \emptyset$, for all $n \geq n_0$. It follows that $(\zeta_n)_{n\in\mathbb{N}}$ is a Weyl sequence for D_2' and λ, and in particular, $\lambda \in \sigma_W(D_2')$. ∎

Proof of Theorem 3.1: Follows from Theorem 3.15 and Proposition 1.1. ∎

Proof of Theorem 3.2: From (1.2), we have that

$$\lambda_0(D_1^{(F)}) = \inf_{\eta \in \mathcal{D}(D_1) \smallsetminus \{0\}} \mathcal{R}_{D_1}(\eta).$$

In particular, it follows that there exists a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D_1) \smallsetminus \{0\}$, such that $\mathcal{R}_{D_1}(\eta_n) \leq \lambda_0(D_1^{(F)}) + 1/n$, for any $n \in \mathbb{N}$. From Proposition 3.14, there exists a sequence $(\zeta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D_2) \smallsetminus \{0\}$, such that $\mathcal{R}_{D_2}(\zeta_n) \leq \lambda_0(D_1^{(F)}) + 2/n$ and $\operatorname{supp} \zeta_n \cap \operatorname{supp} \zeta_{n'} = \emptyset$, for all $n, n' \in \mathbb{N}$, with $n \neq n'$. Evidently, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $\mathcal{R}_{D_2}(\zeta_n) < \lambda_0(D_1^{(F)}) + \varepsilon$, for all $n \geq n_0$. Consider the subspace $\mathcal{H}_\varepsilon$ of $\mathcal{D}(D_2)$ spanned by $\{\zeta_n : n \geq n_0\}$. Since the sections ζ_n, with $n \in \mathbb{N}$, have disjoint supports, the space $\mathcal{H}_\varepsilon$ is infinite dimensional. Clearly, any $\theta \in \mathcal{H}_\varepsilon$ is of the form $\theta := \sum_{n=n_0}^{n_0+\mu} m_n \zeta_n$, for some $\mu \in \mathbb{N}$ and $m_{n_0}, \dots, m_{n_0+\mu} \in \mathbb{F}$. Therefore, we have

$$\mathcal{R}_{D_2}(\theta) = \frac{\sum_{n=n_0}^{n_0+\mu} |m_n|^2 \langle D_2\zeta_n, \zeta_n \rangle_{L^2(E_2)}}{\sum_{n=n_0}^{n_0+\mu} |m_n|^2 \|\zeta_n\|^2_{L^2(E_2)}} \leq \max_{n_0 \leq n \leq n_0+\mu} \mathcal{R}_{D_2}(\zeta_n) < \lambda_0(D_1^{(F)}) + \varepsilon.$$

From Proposition 1.4, it follows that $\lambda_0^{\mathrm{ess}}(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$. ∎

Remark 3.16. In the proof of Theorem 3.2, the only properties of the Friedrichs extension used are self-adjointness and the preservation of the lower bound of D_1, that is, (1.2) holds. Therefore, this proof establishes the analogous result for any self-adjoint extensions of the operators, as long as the extension of D_1 preserves its lower bound.

Corollary 3.17. *Let $p : M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering of manifolds without boundary. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\lambda_0(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$. If, in addition, M_1 is complete, then $\sigma(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$.*

Proof: Follows from Theorems 2.2, 3.1, 3.2, Proposition 2.5 and Remark 2.6. ∎

Corollary 3.18. *Let $p\colon M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering of manifolds with boundary. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\lambda_0^N(S_1) = \lambda_0^{N,\mathrm{ess}}(S_2)$ and $\lambda_0^D(S_1) = \lambda_0^{D,\mathrm{ess}}(S_2)$. If, in addition, M_1 is complete, then $\sigma^D(S_1) \subset \sigma^{D,\mathrm{ess}}(S_2)$.*

Proof: Follows from Theorems 2.2, 3.1, 3.2, Proposition 2.5 and Remark 2.6. ∎

Evidently, in the above corollary, the corresponding inclusion of the Neumann spectra holds, as long as S_1 considered as in (2.2) is essentially self-adjoint. It is not clear if this always holds, provided that M_1 is complete. However, it is easy to see that if V is bounded, then S_1 is essentially self-adjoint, since the Laplacian is essentially self-adjoint (cf. [24, Chapter 8]).

3.2 Finite sheeted coverings

In this section, for sake of completeness, we present the analogous results for finite sheeted coverings. It is clear that they cannot be improved in order to obtain as strong statements as in the case of infinite sheeted amenable coverings.

Proposition 3.19. *Let D_2' be a closed extension of D_2. If p is a finite sheeted Riemannian covering, then $\sigma_{\mathrm{ap}}(\overline{D}_1) \subset \sigma_{\mathrm{ap}}(D_2')$ and $\sigma_W(\overline{D}_1) \subset \sigma_W(D_2')$.*

Proof: If η is in the domain of D_1, then its lift is in the domain of D_2. Consider $\lambda \in \sigma_W(\overline{D}_1)$. From Lemma 1.2, there exists a Weyl sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D_1)$ for $\overline{D}_1$ and λ. From Remark 3.12, it follows that the sequence consisting of the normalized (in $L^2(E_2)$) lifts of η_n, $n \in \mathbb{N}$, is a Weyl sequence for D_2' and λ. Hence, $\sigma_W(\overline{D}_1) \subset \sigma_W(D_2')$. Similarly, it follows that $\sigma_{\mathrm{ap}}(\overline{D}_1) \subset \sigma_{\mathrm{ap}}(D_2')$. ∎

Proposition 3.20. *Assume that D_i is symmetric and bounded from below, and denote by $D_i^{(F)}$ its Friedrichs extension, $i = 1, 2$. If p is a finite sheeted Riemannian covering, then $\lambda_0(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$.*

Proof: If η is in the domain of D_1, then its lift θ is in the domain of D_2. If $\eta \neq 0$, from Remark 3.12, it is easy to see that $\mathcal{R}_{D_1}(\eta) = \mathcal{R}_{D_2}(\theta)$. The statement follows from Proposition 1.5. ∎

The following corollaries describe the behavior of the spectrum of Schrödinger operators under finite sheeted coverings.

Corollary 3.21. *Let $p\colon M_2 \to M_1$ be a finite sheeted Riemannian covering of manifolds without boundary. Consider a Schrödinger operator S_1 on M_1 and let S_2 be its lift on M_2. Then $\lambda_0(S_1) = \lambda_0(S_2)$. If, in addition, M_1 is complete, then $\sigma(S_1) \subset \sigma(S_2)$ and $\sigma_{\mathrm{ess}}(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$.*

Proof: Follows from Propositions 1.1, 3.19, 3.20, 2.5 and Remark 2.6. ∎

For similar reasons, the corresponding statement for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary also holds.

Corollary 3.22. *Let $p\colon M_2 \to M_1$ be a finite sheeted Riemannian covering of complete manifolds without boundary. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\lambda_0^{\mathrm{ess}}(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$ and in particular, $\sigma_{\mathrm{ess}}(S_1) \neq \emptyset$ if and only if $\sigma_{\mathrm{ess}}(S_2) \neq \emptyset$.*

Proof: Follows from Corollary 3.21 and Proposition 2.14. ∎

3.3 Infinite deck transformations group

Let M be a Riemannian manifold, $E \to M$ a Riemannian or Hermitian vector bundle, endowed with a (not necessarily metric) connection ∇ and $D\colon \Gamma(E) \to \Gamma(E)$ a differential operator on E. If M has empty boundary, set $\mathcal{D}(D) := \Gamma_c(E)$. If M has non-empty boundary, consider

$$\mathcal{D}(D) := \{\eta \in \Gamma_c(E) : a\nabla_\nu \eta + b\eta = 0 \text{ on } \partial M\},$$

where ν is the inward pointing normal to ∂M and a, b are real or complex valued functions (depending on whether E is Riemannian or Hermitian) defined on ∂M. It is worth to point out that we do not impose any assumptions[1] on a and b. From Lemma 1.18, the operator D is closable and we denote by $\overline{D}$ its closure.

Theorem 3.23. *Let Γ be a group of automorphisms of E preserving the metric of E, such that the induced action on M is isometric and $D(g_*\eta) = g_* D\eta$, for any $g \in \Gamma$ and $\eta \in \Gamma(E)$. If M has non-empty boundary, assume that ∇, a and b are Γ-invariant along the boundary. If for any compact $K \subset M$, there exists $g \in \Gamma$, such that $gK \cap K = \emptyset$, then $\sigma_{\mathrm{ap}}(\overline{D}) = \sigma_W(\overline{D})$ and $\overline{D}$ does not have eigenvalues of finite multiplicity.*

[1] If D is of order one, then we require $a = 0$.

Proof: Let $\lambda \in \sigma_{\text{ap}}(\overline{D})$. From Lemma 1.2, there exists $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D)$, such that $\|\eta_n\|_{L^2(E)} = 1$ and $(D-\lambda)\eta_n \to 0$ in $L^2(E)$. Since η_n is compactly supported, there exists an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of M, consisting of compact subsets of M, such that $\operatorname{supp}\eta_n \subset K_n$, for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, consider $g_n \in \Gamma$, such that $g_n K_n \cap K_n = \emptyset$, and set $\zeta_n := (g_n)_\star \eta_n$. Then $\zeta_n \in \Gamma_c(E)$ and if M has non-empty boundary, then ζ_n satisfies the same boundary conditions with η_n, since via isometries the boundary is mapped to itself and so does the inward pointing normal. It follows that $\zeta_n \in \mathcal{D}(D)$, $\|\zeta_n\|_{L^2(E)} = 1$ and $(D-\lambda)\zeta_n \to 0$ in $L^2(E)$. Clearly, $\operatorname{supp}\zeta_n = g_n(\operatorname{supp}\eta_n)$, which yields that for any compact $K \subset M$, there exists $n_0 \in \mathbb{N}$, such that $\operatorname{supp}\zeta_n \cap K = \emptyset$, for all $n \geq n_0$. This implies that $\zeta_n \rightharpoonup 0$ in $L^2(E)$, that is, $(\zeta_n)_{n\in\mathbb{N}}$ is a Weyl sequence for $\overline{D}$ and λ. Hence, $\lambda \in \sigma_W(\overline{D})$.

Assume that there exists an eigenvalue λ of $\overline{D}$ of finite multiplicity, and consider $\theta \in \mathcal{D}(\overline{D})$ with $\|\theta\|_{L^2(E)} = 1$ and $\overline{D}\theta = \lambda\theta$. Then there exists a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D)$, such that $\eta_n \to \theta$ and $D\eta_n \to \overline{D}\theta$. Clearly, for $g \in \Gamma$, we have $g_*\eta_n \in \mathcal{D}(D)$, $g_*\eta_n \to g_*\theta$ and $D(g_*\eta_n) \to g_*(\overline{D}\theta)$, which yields that $g_*\theta \in \mathcal{D}(\overline{D})$ and $\overline{D}(g_*\theta) = \lambda(g_*\theta)$.

Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of M, consisting of compact subsets of M, and consider $(g_n)_{n\in\mathbb{N}} \subset \Gamma$, such that $g_n K_n \cap K_n = \emptyset$, for any $n \in \mathbb{N}$. It is clear that the sections $\theta_n := (g_n)_*\theta$ satisfy $\overline{D}\theta_n = \lambda\theta_n$ and $\|\theta_n\|_{L^2(E)} = 1$, for all $n \in \mathbb{N}$. Since the eigenspace corresponding to λ is finite dimensional, after passing to a subsequence, we may assume that $\theta_n \to \theta_0$ in $L^2(E)$, for some θ_0, with $\|\theta_0\|_{L^2(E)} = 1$. Consider a non-zero $\zeta \in \Gamma_c(E)$ and set $\zeta_n := (g_n^{-1})_*\zeta$. Then

$$\langle \theta_n, \zeta \rangle^2_{L^2(E)} = \langle \theta, \zeta_n \rangle^2_{L^2(E)} \leq \|\zeta\|^2_{L^2(E)} \int_{\operatorname{supp}\zeta_n} \|\theta\|^2.$$

Let $\varepsilon > 0$ and consider a compact $K \subset M$, such that $\int_{M\smallsetminus K}\|\theta\|^2 < \varepsilon^2/\|\zeta\|^2_{L^2(E)}$. Since $\operatorname{supp}\zeta$ and K are eventually subsets of K_n, there exists $n_0 \in \mathbb{N}$, such that $\operatorname{supp}\zeta_n \cap K = \emptyset$, for all $n \geq n_0$. Therefore, for $n \geq n_0$, we have $\operatorname{supp}\zeta_n \subset M \smallsetminus K$, and in particular, $|\langle \theta_n, \zeta \rangle_{L^2(E)}| < \varepsilon$. This yields that $\theta_n \rightharpoonup 0$ in $L^2(E)$, which is a contradiction, since $\theta_n \to \theta_0$ in $L^2(E)$ and $\|\theta_0\|_{L^2(E)} = 1$. ∎

Theorem 3.24. *Assume that D is symmetric and bounded from below, and denote by $D^{(F)}$ its Friedrichs extension. Under the assumptions of Theorem 3.23, the spectrum of $D^{(F)}$ is essential and $D^{(F)}$ does not have eigenvalues of finite multiplicity.*

Proof: Let $\eta \in \mathcal{D}(D^{(F)})$ and $g \in \Gamma$. From the invariance of $\mathcal{D}(D)$ and D under the action of Γ, it follows that $g_*\eta \in \mathcal{D}(D^{(F)})$ and $D^{(F)}(g_*\eta) = g_*(D^{(F)}\eta)$. As in the proof of Theorem 3.23, it follows that $D^{(F)}$ does not have eigenvalues of finite multiplicity. From Proposition 1.1, we obtain that $\sigma(D^{(F)}) = \sigma_{\text{ess}}(D^{(F)})$. ∎

The above theorems can be applied to Riemannian coverings with infinite deck transformations group. In the context described in the beginning of this chapter, we obtain the following consequences.

Corollary 3.25. *If the deck transformations group of the covering is infinite, then $\overline{D}_2$ does not have eigenvalues of finite multiplicity and $\sigma_{\mathrm{ap}}(\overline{D}_2) = \sigma_W(\overline{D}_2)$.*

Proof: Follows immediately from Theorem 3.23, for Γ being the deck transformations group of the covering. ∎

Corollary 3.26. *Assume that D_2 is essentially self-adjoint. If the deck transformations group of the covering is infinite, then $\overline{D}_2$ does not have eigenvalues of finite multiplicity and in particular, $\sigma(\overline{D}_2) = \sigma_{\mathrm{ess}}(\overline{D}_2)$.*

Proof: Follows from Corollary 3.25 and Proposition 1.1. ∎

Corollary 3.27. *Assume that D_2 is symmetric and bounded from below, and denote by $D_2^{(F)}$ its Friedrichs extension. If the deck transformations group of the covering is infinite, then $D_2^{(F)}$ does not have eigenvalues of finite multiplicity and $\sigma(D_2^{(F)}) = \sigma_{\mathrm{ess}}(D_2^{(F)})$.*

Proof: Follows from Theorem 3.24, for Γ being the deck transformations group of the covering. ∎

Corollary 3.28. *Let $p \colon M_2 \to M_1$ be a Riemannian covering with infinite deck transformations group. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\sigma^N(S_2) = \sigma^N_{\mathrm{ess}}(S_2)$.*

Proof: Follows from Corollary 3.27. ∎

In the above corollary, manifolds are not required to have non-empty boundary. In particular, the corresponding statement holds for operators on manifolds without boundary. Moreover, the manifolds are not required to be complete. Therefore, from Remark 2.1, the corresponding statement also holds for the Dirichlet spectrum of operators on manifolds with boundary.

Corollary 3.29. *Let M be a Riemannian manifold without boundary, and assume that there exists a non-zero $\lambda_0(M)$-harmonic function in $L^2(M)$. If Γ is a discrete group acting freely and properly discontinuously on M via isometries, then Γ is finite.*

Proof: From [22, Theorem 2.8], if $\lambda_0(M)$ is an eigenvalue, then the corresponding eigenspace is one dimensional. The statement follows from Corollary 3.26. ∎

Besides Riemannian coverings, the above theorems can be applied to manifolds with high symmetry. For instance, it follows that the spectrum of the Laplacian on a non-compact homogeneous space is essential. Moreover, we obtain the analogous statement, if there exists a non-compact Lie group acting on the manifold properly discontinuously via isometries.

3.4 Applications and examples

The following application of our results is motivated by Corollary 3.8 of the arXiv version of [1].

Theorem 3.30. *Let $p\colon M_2 \to M_1$ be a Riemannian covering with M_2 simply connected, complete and without boundary. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. If there exists a compact $K \subset M_1$, such that the image of the fundamental group of any connected component of $M_1 \smallsetminus K$ in $\pi_1(M_1)$ is amenable, then $\sigma_{\mathrm{ess}}(S_1) \subset \sigma_{\mathrm{ess}}(S_2)$.*

Proof: Let $\lambda \in \sigma_{\mathrm{ess}}(S_1)$. From Proposition 2.13, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)$, such that $\|f_n\|_{L^2(M_1)} = 1$, $(S-\lambda)f_n \to 0$ in $L^2(M_1)$ and for every compact subset K_0 of M_1, there exists $n_0 \in \mathbb{N}$, such that $\operatorname{supp} f_n \cap K_0 = \emptyset$, for all $n \geq n_0$. Without loss of generality, we may assume that the supports of f_n are connected, since we may restrict each f_n to a connected component of its support and obtain a sequence with the same properties.

Consider a compact $K \subset M_1$, such that the image of the fundamental group of any connected component of $M_1 \smallsetminus K$ in $\pi_1(M_1)$ is amenable. Clearly, after passing to a subsequence, we may assume that the functions f_n are supported in $M_1 \smallsetminus K$. Since $\operatorname{supp} f_n$ is connected, for any $n \in \mathbb{N}$, it follows that $\operatorname{supp} f_n \subset U_n$, where U_n is a connected component of $M_1 \smallsetminus K$. From the Lifting Theorem, it follows that the inclusion $U_n \hookrightarrow M_1$ can be lifted to the covering space $M_n' := M_2/\Gamma_n$, where Γ_n is the image of $\pi_1(U_n)$ in $\pi_1(M_1)$. In particular, any f_n can be lifted to some $f_n' \in C_c^\infty(M_n')$.

Since the covering $q_n\colon M_2 \to M_n'$ is normal with deck transformations group Γ_n, it follows that it is amenable. If q_n is finite sheeted, let $\tilde{f}_n$ be the normalized (in $L^2(M_2)$) lift of f_n' on M_2. If q_n is infinite sheeted, from Proposition 3.13, there exists $\tilde{f}_n \in C_c^\infty(M_2)$, such that $\|\tilde{f}_n\|_{L^2(M_2)} = 1$, $\operatorname{supp} \tilde{f}_n \subset q_n^{-1}(\operatorname{supp} f_n')$ and

$$\|(S_2-\lambda)\tilde{f}_n\|_{L^2(M_2)} \leq \|(S_n'-\lambda)f_n'\|_{L^2(M_n')} + \frac{1}{n} = \|(S_1-\lambda)f_n\|_{L^2(M_1)} + \frac{1}{n},$$

where S_n' is the lift of S_1 on M_n'. Therefore, $(S_2-\lambda)\tilde{f}_n \to 0$ in $L^2(M_2)$ and $\operatorname{supp} \tilde{f}_n$ is contained in $p^{-1}(\operatorname{supp} f_n)$. From Proposition 2.13, it follows that $\lambda \in \sigma_{\mathrm{ess}}(S_2)$. ∎

Remark 3.31. In the proof of Theorem 3.30, the only properties of Schrödinger operators used are essential self-adjointness and Proposition 2.13, which follows from the Decomposition Principle. Therefore, this proof establishes the analogous result for essentially self-adjoint differential operators, for which the Decomposition Principle holds (cf. [4]). For instance, if M_1 has empty boundary, then the statement of Theorem 3.30 holds for any elliptic differential operator D_1, such that D_1 and D_2 are essentially self-adjoint on the spaces of compactly supported smooth sections.

Corollary 3.32. *Let $p\colon M_2 \to M_1$ be a Riemannian covering with M_1 complete and without boundary. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0(S_1) \in \sigma_{\text{ess}}(S_1)$, and S_2 its lift on M_2. If there exists a compact $K \subset M_1$, such that the image of the fundamental group of any connected component of $M_1 \smallsetminus K$ in $\pi_1(M_1)$ is amenable, then $\lambda_0(S_1) = \lambda_0(S_2)$.*

Proof: Follows immediately from Theorem 3.30, Proposition 2.5 and Remark 2.6. ∎

Let $p\colon M_2 \to M_1$ be a Riemannian covering of complete manifolds. As remarked in [8], there exists a non-amenable Riemannian covering that preserves the bottom of the spectrum of the Laplacian. From Theorem 3.1, Propositions 3.19 and 1.1, if p is amenable, then $\sigma(M_1) \subset \sigma(M_2)$. It is natural to examine if this inclusion implies amenability of the covering. From Theorem 3.30, it is easy to construct an example of a non-amenable, normal Riemannian covering $p\colon M_2 \to M_1$ with M_1 complete, with bounded geometry and of finite topological type (that is, M_1 admits a finite triangulation, where the simplices are defined on the standard simplex with possibly some lower dimensional faces removed), such that $\sigma(M_1) = \sigma(M_2)$.

Example 3.33. Let M_1 be a two dimensional torus with a cusp, endowed with a Riemannian metric, such that M_1 is complete and outside a compact set the metric is the standard metric of the flat cylinder. Evidently, M_1 is of finite topological type and has bounded geometry. Moreover, we have that $\sigma_{\text{ess}}(M_1) = [0, +\infty)$ (cf. [17, Theorem 1]). Clearly, there exists a compact subset K of M_1, such that $\pi_1(M_1 \smallsetminus K) = \mathbb{Z}$. From Theorem 3.30, it follows that for the simply connected covering space M_2 of M_1, we have $\sigma_{\text{ess}}(M_2) = [0, +\infty)$. However, $\pi_1(M_1)$ is the free group with two generators, which is non-amenable (from Example 1.14).

The next observation, provides a sufficient geometric condition for amenability of coverings.

Proposition 3.34. *Let M_1 be a complete Riemannian manifold, without boundary and with non-negative Ricci curvature. Then any covering $p\colon M_2 \to M_1$ is amenable.*

Proof: Let $\tilde{M}$ be the simply connected covering space of M_1. From the Bishop-Gromov Comparison Theorem, it follows that $\tilde{M}$ has polynomial growth and hence, every finitely generated subgroup of $\pi_1(M_1)$ has polynomial growth (cf. [19]). From Corollary 1.11, it follows that every finitely generated subgroup of $\pi_1(M_1)$ is amenable and Corollary 1.12 yields that so is $\pi_1(M_1)$. Therefore, any covering $p\colon M_2 \to M_1$ is amenable. ∎

Next, we present an example of an infinite sheeted amenable covering with trivial deck transformations group. In particular, this implies that the results of Section 3.3 cannot be applied to arbitrary infinite sheeted amenable coverings.

Example 3.35. Let Γ_1 be the countable group of invertible, upper triangular 2×2 matrices with entries in $\mathbb{Q}$ and let M_1 be a Riemannian manifold with $\pi_1(M_1) = \Gamma_1$ (cf. [3, Section 5]). Let Γ_2 be the subgroup of Γ_1 consisting of diagonal matrices. Denote by $\tilde{M}$ the simply connected covering space of M_1 and consider $M_2 := \tilde{M}/\Gamma_2$. It is easy to see that the covering $p\colon M_2 \to M_1$ is infinite sheeted and does not have non-trivial deck transformations. However, Γ_1 is solvable and in particular, amenable (from Corollary 1.13), which yields that p is an amenable covering.

Recall that in the main results of this chapter there are no assumptions on the vector bundles, the connections and the differential operators. We end this section with an example which demonstrates that these play a crucial role in the behavior of the spectrum even under finite sheeted coverings. Namely, this example shows that whether or not the bottom of the spectrum of the connection Laplacian is preserved under a Riemannian covering depends on the corresponding metric connection.

If M is a closed Riemannian manifold and $E \to M$ is a Riemannian vector bundle endowed with a metric connection ∇, then the (corresponding) connection Laplacian is given by $\Delta = \nabla^* \nabla$. It is well-known that $\Delta\colon \Gamma(E) \subset L^2(E) \to L^2(E)$ is essentially self-adjoint and its spectrum is discrete (cf. [16]).

Example 3.36. Consider $S^1 := \mathbb{R}/\mathbb{Z}$ and the trivial bundle $E_1 := S^1 \times \mathbb{R}^2$ with the standard metric. We can identify smooth sections of E_1 with smooth maps $f\colon \mathbb{R} \to \mathbb{R}^2$ with $f(x) = f(x+1)$, for all $x \in \mathbb{R}$. For $\phi \in \mathbb{R}$, consider the metric connection ∇^ϕ, defined by

$$\nabla^\phi_{\frac{d}{dx}} f(x) := \begin{pmatrix} \cos(x\phi) & -\sin(x\phi) \\ \sin(x\phi) & \cos(x\phi) \end{pmatrix} \frac{d}{dx} \begin{pmatrix} \cos(x\phi) & \sin(x\phi) \\ -\sin(x\phi) & \cos(x\phi) \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

for any smooth section $f = (f_1, f_2)$ of E_1. Since the spectrum of the connection Laplacian Δ^ϕ is discrete for any $\phi \in \mathbb{R}$, it is clear that $\lambda_0(\Delta^\phi, E_1) = 0$ if and only if there exists a parallel section of E_1 with respect to ∇^ϕ, or equivalently, $\phi = 2k\pi$, for some $k \in \mathbb{Z}$.

For $q \in \mathbb{N} \smallsetminus \{1\}$, consider a q-sheeted Riemannian covering of S_1 and the pullback bundle E_2 of E_1 endowed with the standard metric and the pullback connection ∇^ϕ. It is clear that $\lambda_0(\Delta^{2\pi}, E_2) = \lambda_0(\Delta^{2\pi}, E_1) = 0$. However, the above arguments imply that $\lambda_0(\Delta^{2\pi/q}, E_2) = 0 < \lambda_0(\Delta^{2\pi/q}, E_1)$.

CHAPTER 4

Coverings preserving the bottom of the spectrum

In the previous chapter we studied the behavior of the spectrum under amenable coverings. In particular, we proved that amenable coverings preserve the bottom of the spectrum of Schrödinger operators. In this chapter we examine to what extent the converse implication holds. In particular, the aim of this chapter is to prove the following result.

Theorem 4.1. *Let $p\colon M_2 \to M_1$ be a Riemannian covering. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$, and S_2 its lift on M_2. Then $\lambda_0(S_2) = \lambda_0(S_1)$ if and only if p is amenable.*

4.1 Manifolds with Ricci curvature bounded from below

In this subsection we recall the main result of [2] and point out that its proof, with some slight modifications, establishes this result for possibly non-connected covering spaces.

A non-connected Riemannian manifold M is complete if all of its connected components are complete. The distance between points of different connected components of M is considered to be infinite. In particular, any bounded subset of M is contained in a connected component of M.

Theorem 4.2 ([2, Theorem 4.1]). *Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_2 possibly non-connected. Assume that M_1 is complete, without boundary, and with Ricci curvature bounded from below. Let $S_1 = \Delta + V$ be a Schrödinger operator on M_1, with V*

and grad V *bounded, and let* S_2 *be its lift on* M_2*. If* $\lambda_0(S_2) = \lambda_0(S_1) \neq \lambda_0^{\text{ess}}(S_1)$*, then the covering is amenable.*

We begin with some definitions and remarks from [2]. Let M be a possibly non-connected Riemannian manifold. A positive $\varphi \in C^\infty(M)$ *satisfies a Harnack estimate* if there exists a constant $c_\varphi \geq 1$, such that

$$\sup_{B(x,r)} \varphi^2 \leq c_\varphi^{r+1} \inf_{B(x,r)} \varphi^2,$$

for all $x \in M$ and $r > 0$. Assume that M is complete, with Ricci curvature bounded from below, and let $S = \Delta + V$ be a Schrödinger operator on M, with V and grad V bounded. From [10, Theorem 6], if a positive $\varphi \in C^\infty(M)$ satisfies $S\varphi = \lambda\varphi$, for some $\lambda \in \mathbb{R}$, then φ satisfies a Harnack estimate.

The *modified Cheeger's constant* of M is defined as

$$h_\varphi(M) := \inf_A \frac{\int_{\partial A} \varphi^2}{\int_A \varphi^2},$$

where the infimum is taken over all smoothly bounded compact domains A of M.

Lemma 4.3. *Let M be a possibly non-connected, complete Riemannian manifold, without boundary and with Ricci curvature bounded from below. Let $\varphi \in C^\infty(M)$ be a positive function, which satisfies a Harnack estimate. If $h_\varphi(M) = 0$, then for any $\varepsilon, r > 0$, there exists a bounded open subset A of M, such that*

$$\int_{A^r \smallsetminus A} \varphi^2 < \varepsilon \int_A \varphi^2,$$

where $A^r := \{y \in M : d(y, A) < r\}$.

Proof: We may renormalize the Riemannian metric of M, so that $\mathrm{Ric}_M \geq 1 - m$, where m is the dimension of M. Since $h_\varphi(M) = 0$, for any $\varepsilon, r > 0$, there exists a non-empty, bounded domain A of M satisfying the estimate (3.2) of [2]. Evidently, A is contained in a connected component of M and the arguments of the proof of [2, Lemma 3.1] can be carried out in this connected component of M, establishing the asserted claim. ∎

Lemma 4.4. *In the setting of Theorem 4.2, there exists a compact $K \subset M_1$, such that for any $\varepsilon, r > 0$, there exists $z \in K$ and a bounded open subset A of M_2, such that*

$$\#(p^{-1}(z) \cap (A^r \smallsetminus A)) < \varepsilon \#(p^{-1}(z) \cap A).$$

Proof: Since $\lambda_0(S_1) \notin \sigma_{\mathrm{ess}}(S_1)$, from Proposition 2.14, there exists a compact subset K of M_1, such that $\lambda_0(S_1, M_1 \smallsetminus K) > \lambda_0(S_1)$. The proof is identical to the one of [2, Lemma 4.5], taking into account that [2, Lemma 3.1] has been extended to possibly non-connected manifolds in Lemma 4.3. ∎

Proof of Theorem 4.2: Fix $x \in M_1$ and consider the fundamental group $\pi_1(M_1)$ with base point x. Consider a compact set $K \subset M_1$ as in Lemma 4.4, and let $R > 0$, such that $K \subset B(x, R)$. Let $\varepsilon > 0$ and let G be a finite subset of $\pi_1(M_1)$. For each $g \in G$, consider a smooth representative loop γ_g with base point x, and consider

$$r > \max_{g \in G} \ell(\gamma_g) + 2R,$$

where $\ell(\cdot)$ stands for the length of a curve. From Lemma 4.4, there exists $z \in K$ and a bounded open subset A of M_2, such that

$$\#(p^{-1}(z) \cap (A^r \smallsetminus A)) < \varepsilon \#(p^{-1}(z) \cap A).$$

Consider a smooth path $\gamma\colon [0,1] \to M_1$ from x to z, of length less than R. For $y \in p^{-1}(x)$, let $\tilde{\gamma}\colon [0,1] \to M_2$ be the lift of γ with $\tilde{\gamma}(0) = y$, and set $\Phi(y) := \tilde{\gamma}(1)$. Then the map $\Phi\colon p^{-1}(x) \to p^{-1}(z)$ is bijective. Let $F := \Phi^{-1}(p^{-1}(z) \cap A)$ and consider $y \in F \smallsetminus Fg$, for some $g \in G$. Then $\Phi(y) \in A$ and $\Phi(y \cdot g^{-1}) \notin A$. Evidently, we have

$$d(\Phi(y), \Phi(y \cdot g^{-1})) \leq d(y, y \cdot g^{-1}) + 2\ell(\gamma) \leq \ell(\gamma_g) + 2R < r.$$

Therefore, $\Phi(y \cdot g^{-1}) \in A^r \smallsetminus A$. Since Φ is bijective, it is clear that

$$\begin{aligned} \#(F \smallsetminus Fg) &= \#\{y \cdot g^{-1} : y \in F \smallsetminus Fg\} = \#\{\Phi(y \cdot g^{-1}) : y \in F \smallsetminus Fg\} \\ &\leq \#(p^{-1}(z) \cap (A^r \smallsetminus A)) < \varepsilon \#(p^{-1}(z) \cap A) = \varepsilon \#(F). \end{aligned}$$

From Proposition 1.9, it follows that the covering is amenable. ∎

4.2 Coverings of compact manifolds

Throughout this section, for simplicity of notation, we denote by $\mathcal{R}(f)$ the Rayleigh quotient of a Lipschitz function f with respect to the Laplacian. Essentially, in this subsection we prove an analogue of Brooks' result [9] involving the bottom of the Neumann spectrum of the Laplacian. The fact that amenable Riemannian coverings preserve the bottom of the Neumann spectrum has been established in the previous chapter. Therefore, it remains to prove the converse implication, in case the base manifold is compact. For reasons that will become clear in the next section, we need to establish it for possibly non-connected covering spaces.

Theorem 4.5. *Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_1 compact with boundary, and M_2 possibly non-connected. If $\lambda_0^N(M_2) = 0$, then p is amenable.*

Let ν_i be the inward pointing normal to ∂M_i, $i = 1,2$. Then there exists $\delta > 0$, such that the map $\Phi\colon \partial M_1 \times [0,2\delta) \to M_1$, defined by $\Phi(x,t) := \exp_x(t\nu_1)$, is a diffeomorphism onto its image. By definition, any point of M_1 has an evenly covered neighborhood with respect to the restriction of p on any connected component of M_2. Therefore, we may assume that δ is sufficiently small, so that for any $x \in \partial M_1$ and $y_1, y_2 \in p^{-1}(x)$, with $y_1 \neq y_2$, we have $d(y_1,y_2) \geq 2\delta$. It is worth to point out that we consider the distance between points of different connected components of M_2 to be infinite.

Lemma 4.6. *The map $\Psi\colon \partial M_2 \times [0,\delta) \to M_2$, defined by $\Psi(y,t) := \exp_y(t\nu_2)$, is a diffeomorphism onto its image.*

Proof: Since $(p \circ \Psi)(y,t) = \Phi(p(y),t)$, for any $y \in \partial M_2$ and $t \in [0,\delta)$, it is clear that Ψ is a local diffeomorphism. So, it suffices to prove that it is injective. Consider $y_1, y_2 \in \partial M_2$ and $t_1, t_2 \in [0,\delta)$, such that $\Psi(y_1,t_1) = \Psi(y_2,t_2) =: z$. Evidently, $d(y_i,z) < \delta$, $i = 1,2$, which yields that $d(y_1,y_2) < 2\delta$. Moreover, it follows that $\Phi(p(y_1),t_1) = \Phi(p(y_2),t_2)$. Since Φ is a diffeomorphism onto its image, this yields that $t_1 = t_2$, $p(y_1) = p(y_2)$, and in particular, $y_1 = y_2$. ∎

Lemma 4.7. *There exists a Riemannian metric g' on M_1, such that the map Φ restricted on $\partial M_1 \times [0,\delta)$ is an isometry onto its image.*

Proof: Let g_c be the push-forward of the product metric of $\partial M_1 \times [0,2\delta)$ via Φ. Denote by g the original Riemannian metric of M_1. Consider a smooth function $\tau\colon [0,2\delta) \to [0,1]$, with $\tau(t) = 1$ for $t \leq \delta$, and $\tau(t) = 0$ for $t \geq 3\delta/2$. Consider the function $\tau' \in C^\infty(M_1)$, defined by $\tau'(\Phi(x,t)) = \tau(t)$ in $\Phi(\partial M_1 \times [0,2\delta))$, and $\tau' = 0$ otherwise. Evidently, the Riemannian metric

$$g' := \tau' g_c + (1-\tau')g.$$

on M_1 satisfies the desired property. ∎

Consider M_1 and M_2 endowed with g' and its lift, respectively. Evidently, since $(p \circ \Psi)(y,t) = \Phi(p(y),t)$, for any $y \in \partial M_2$ and $t \in [0,\delta)$, it follows that Ψ restricted on $\partial M_2 \times [0,\delta)$ is a local isometry, with respect to the lift of g'. From Lemma 4.6, this map is also injective, which yields that it is an isometry onto its image. Denote by U_t the open set $\Psi(\partial M_2 \times [0,t))$, and by C_t the closed set $\Psi(\partial M_2 \times \{t\})$.

Evidently, there exist $c_1, c_2 > 0$, such that for any $f \in C^\infty(M_2)$, the norms of the gradients of f with respect to the lifts of g and g', are related by

$$c_1 \|\operatorname{grad}_g f\|_g \leq \|\operatorname{grad}_{g'} f\|_{g'} \leq c_2 \|\operatorname{grad}_g f\|_g.$$

Moreover, there exists a positive, smooth $\mathcal{V}\colon M_1 \to \mathbb{R}$, such that the volume elements induced by the lifts of g and g′, satisfy

$$dVol_{g'} = (\mathcal{V} \circ p) dVol_g.$$

Therefore, for any non-zero $f \in C_c^\infty(M_2)$, the Rayleigh quotients of f with respect to the Laplacians induced by the lifts of g and g′, satisfy

$$\mathcal{R}_{g'}(f) = \frac{\int_{M_2} \| \operatorname{grad}_{g'} f\|_{g'}^2 dVol_{g'}}{\int_{M_2} f^2 dVol_{g'}} \leq c_2^2 \frac{\max \mathcal{V}}{\min \mathcal{V}} \mathcal{R}_g(f).$$

From Proposition 2.4, since $\lambda_0^N(M_2) = 0$ with respect to the lift of g, it follows that $\lambda_0^N(M_2) = 0$ with respect to the lift of g′. From now on, we will be working with g′ and its lift. It is worth to point out that the maps Φ and Ψ are defined in terms of the exponentials with respect to the original Riemannian metrics.

Lemma 4.8. *For any $\varepsilon > 0$, there exists $f \in \operatorname{Lip}_c(M_2)$, smooth on $M_2 \smallsetminus C_{t_0}$, for one $t_0 \in (0,\delta)$, with $f|_{\partial M_2}$ non-zero, such that $\mathcal{R}(f) \leq \varepsilon$ and*

$$\frac{\int_{\partial M_2} \| \operatorname{grad} f\|^2}{\int_{\partial M_2} f^2} \leq \varepsilon.$$

Proof: Since $\lambda_0^N(M_2) = 0$, from Proposition 2.4, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, such that $\mathcal{R}(f_n) \to 0$. Assume that there exists $\varepsilon > 0$, such that for any $n \in \mathbb{N}$ and $t \in [0,\delta)$, we have

$$\int_{C_t} \| \operatorname{grad} f_n\|^2 > \varepsilon \int_{C_t} f_n^2. \tag{4.1}$$

Then

$$\int_{U_\delta} \| \operatorname{grad} f_n\|^2 > \varepsilon \int_{U_\delta} f_n^2,$$

which yields that $\int_{U_\delta} f_n^2 \to 0$ and $\int_{M_2 \smallsetminus U_\delta} f_n^2 \to 1$. Let $\chi \in C^\infty(M_1)$, with $\chi(x) = 1$ for $d(x,\partial M_1) \geq \delta$, and $\chi(x) = 0$ for $d(x, \partial M_1) < \delta/2$. Let $\tilde{\chi} \in C^\infty(M_2)$ be the lift of χ. Then $\tilde{\chi} = 0$ in $U_{\delta/2}$ and $\tilde{\chi} = 1$ outside U_δ. For $g_n := \tilde{\chi} f_n \in C_c^\infty(M_2)$, we have

$$\|g_n\|_{L^2(M_2)}^2 = \int_{U_\delta} \tilde{\chi}^2 f_n^2 + \int_{M_2 \smallsetminus U_\delta} f_n^2 \to 1,$$

and

$$\int_{M_2} \| \operatorname{grad} g_n\|^2 \leq 2 \int_{U_\delta} (\tilde{\chi}^2 \| \operatorname{grad} f_n\|^2 + f_n^2 \| \operatorname{grad} \tilde{\chi}\|^2) + \int_{M_2 \smallsetminus U_\delta} \| \operatorname{grad} f_n\|^2 \to 0.$$

Therefore, we have that $\mathcal{R}(g_n) \to 0$. Since g_n is supported in the interior of M_2, for any $n \in \mathbb{N}$, from Proposition 2.4 and Remark 2.1, it follows that $\lambda_0^D(M_2) = 0$. This is a contradiction, since from Proposition 2.5, Remarks 2.6 and 2.1, we have $\lambda_0^D(M_2) \geq \lambda_0^D(M_1) > 0$.

Hence, (4.1) cannot hold, that is, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and $t \in [0, \delta)$, such that

$$\int_{C_t} \|\operatorname{grad} f_n\|^2 \leq \varepsilon \int_{C_t} f_n^2. \tag{4.2}$$

Let $0 < \varepsilon < \lambda_0^D(M_2)$ and consider $f_n \in C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, $\mathcal{R}(f_n) < \varepsilon$, satisfying (4.2) for some $t \in [0, \delta)$. Let t_0 be the minimum of all $t \in [0, \delta)$, for which (4.2) holds. If $t_0 = 0$, then f_n is the desired function. Otherwise, define $f \in C_c(M_2)$ by $f = f_n$ outside U_{t_0}, and $f(\Psi(x,t)) = f_n(\Psi(x, t_0))$ for $t \leq t_0$. It is clear that $f \in \operatorname{Lip}_c(M)$ and is smooth on $M_2 \smallsetminus C_{t_0}$.

Since $\mathcal{R}(f_n) < \lambda_0^D(M_2)$, from Proposition 2.4 and Remark 2.1, it follows that f_n is not identically zero on U_{t_0}. Since $\mathcal{R}(f_n) < \varepsilon$, from the definition of t_0, it follows that f_n is not identically zero on $M_2 \smallsetminus U_{t_0}$. In particular, this yields that f is non-zero. Since (4.2) holds for $t = t_0$, we have

$$\int_{\partial M_2} \|\operatorname{grad} f\|^2 = \int_{C_{t_0}} \|\operatorname{grad}(f_n|_{C_{t_0}})\|^2 \leq \varepsilon \int_{C_{t_0}} f_n^2 = \varepsilon \int_{\partial M_2} f^2.$$

Furthermore, we have

$$\begin{aligned} \mathcal{R}(f) &= \frac{\int_0^{t_0} \int_{C_t} \|\operatorname{grad} f\|^2 + \int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_0^{t_0} \int_{C_t} f^2 + \int_{M_2 \smallsetminus U_{t_0}} f_n^2} \\ &\leq \frac{\varepsilon \int_0^{t_0} \int_{C_t} f^2 + \int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_0^{t_0} \int_{C_t} f^2 + \int_{M_2 \smallsetminus U_{t_0}} f_n^2} \\ &\leq \max\left\{ \varepsilon, \frac{\int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_{M_2 \smallsetminus U_{t_0}} f_n^2} \right\}. \end{aligned} \tag{4.3}$$

It is clear that

$$\begin{aligned} \varepsilon &> \mathcal{R}(f_n) = \frac{\int_0^{t_0} \int_{C_t} \|\operatorname{grad} f_n\|^2 + \int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_0^{t_0} \int_{C_t} f_n^2 + \int_{M_2 \smallsetminus U_{t_0}} f_n^2} \\ &\geq \min\left\{ \frac{\int_0^{t_0} \int_{C_t} \|\operatorname{grad} f_n\|^2}{\int_0^{t_0} \int_{C_t} f_n^2}, \frac{\int_{M_2 \smallsetminus U_{t_0}} \|\operatorname{grad} f_n\|^2}{\int_{M_2 \smallsetminus U_{t_0}} f_n^2} \right\}. \end{aligned}$$

From the definition of t_0, the first term is greater than ε, which yields that the second term is smaller than ε. From (4.3), it follows that $\mathcal{R}(f) \leq \varepsilon$. Since $\varepsilon < \lambda_0^D(M_2)$, from Remark 2.1 and Proposition 2.4, it is clear that f cannot vanish identically on ∂M_2. ∎

Glue the cylinder $\partial M_1 \times [0, +\infty)$, with the product metric, along ∂M_1, so that $\partial/\partial t$ is the outward pointing normal to ∂M_1. Denote by N_1 the obtained Riemannian manifold. The covering $p \colon M_2 \to M_1$ can be extended to a Riemannian covering $p \colon N_2 \to N_1$, where N_2 is the Riemannian manifold obtained by gluing $\partial M_2 \times [0, +\infty)$ along ∂M_2 in the analogous way. Evidently, $p \colon M_2 \to M_1$ is amenable if and only if $p \colon N_2 \to N_1$ is amenable. Points in $N_i \smallsetminus M_i^\circ$ will be written in the form (x, t), with $x \in \partial M_i$ and $t \geq 0$, $i = 1, 2$.

Consider a positive smooth $\phi \colon [0, +\infty) \to \mathbb{R}$, with $\phi(t) = 1$ for $t \leq 1/2$, and $\phi(t) = e^{-t}$ for $t \geq 1$. Let $\varphi \in C^\infty(N_1)$ be the square-integrable function defined by $\varphi = 1$ in M_1, and $\varphi(x, t) = \phi(t)$ in $N_1 \smallsetminus M_1$. Consider the function $V \in C^\infty(N_1)$, defined by $V = 0$ in M_1, and $V(x, t) = \phi''(t)/\phi(t)$ in $N_1 \smallsetminus M_1$. It is worth to point out that outside the compact set $M_1 \cup (\partial M_1 \times [0, 1])$, we have that $V = 1$ and in particular, V is bounded from below. Consider the Schrödinger operator $S_1 := \Delta + V$ on N_1 and its lift S_2 on N_2. It is clear that $S_1 \varphi = 0$.

Remark 4.9. Evidently, N_1 is complete, without boundary and with Ricci curvature bounded from below. Since $V = 1$ outside the compact set $M_1 \cup (\partial M_1 \times [0, 1])$, from Propositions 2.14 and 2.4, it follows that $\lambda_0^{\mathrm{ess}}(S_1) \geq 1$. Moreover, it is clear that V and $\operatorname{grad} V$ are bounded.

Lemma 4.10. *The function φ belongs to the domain of the Friedrichs extension of S_1 and in particular, $\lambda_0(S_1) = 0$.*

Proof: For $T > 0$, consider the compactly supported Lipschitz function χ_T defined by $\chi_T = 1$ in M_1, $\chi_T(x, t) = 1$ for $t \leq T$, $\chi_T(x, t) = T + 1 - t$ for $T \leq t \leq T + 1$, and $\chi_T(x, t) = 0$ for $t \geq T$. Then $\chi_T \varphi \in H_0^1(N_1)$ for any $T > 0$, and

$$\| \varphi - \chi_T \varphi \|_{L^2(N_1)}^2 \leq \int_{\partial M_1 \times [T, +\infty)} \varphi^2.$$

Moreover, we have

$$\begin{aligned} \int_{N_1} \| \operatorname{grad}(\varphi - \chi_T \varphi) \|^2 &\leq 2 \int_{N_1} ((1 - \chi_T)^2 \| \operatorname{grad} \varphi \|^2 + \varphi^2 \| \operatorname{grad}(1 - \chi_T) \|^2) \\ &\leq 2 \int_{\partial M_1 \times [T, +\infty)} \| \operatorname{grad} \varphi \|^2 + 2 \int_{\partial M_1 \times [T, T+1]} \varphi^2. \end{aligned}$$

Since $\varphi(x, t) = \| \operatorname{grad} \varphi(x, t) \| = e^{-t}$, for $t \geq 1$, it follows that $\chi_T \varphi \to \varphi$ in $H_0^1(N_1)$, as $T \to +\infty$. Since V is bounded, it follows that $\varphi \in H_V(N_1)$, where H_V is the space defined in Chapter 2. Moreover, since $S_1\varphi = 0$, it follows that φ is an eigenfunction of the Friedrichs extension of S_1 and in particular, $\lambda_0(S_1) \leq 0$. From Proposition 2.7, since φ is positive, it follows that $\lambda_0(S_1) = 0$. ∎

Denote by $\tilde{\varphi}$ the lift of φ on N_2 and consider the renormalization $S_{\tilde{\varphi}}$ of S_2 with respect to $\tilde{\varphi}$. Let $g \in \mathrm{Lip}_c(M_2)$, such that g restricted on ∂M_2 is non-zero and smooth, and $h\colon [0,+\infty) \to \mathbb{R}$ be a compactly supported, smooth function, with $h(t) = 1$ for $t \leq 1/2$. Extend g in the glued ends $\partial M_2 \times [0,+\infty)$ by

$$g(x,t) := g(x)h(t). \tag{4.4}$$

It is clear that $g \in \mathrm{Lip}_c(N_2)$, and in the glued ends, we have

$$\operatorname{grad} g(x,t) = g(x)h'(t)\frac{\partial}{\partial t} + h(t)\operatorname{grad} g(x).$$

In particular, it follows that

$$\|\operatorname{grad} g(x,t)\|^2 = g^2(x)h'(t)^2 + h^2(t)\|\operatorname{grad} g(x)\|^2,$$

which yields that

$$\begin{aligned} \frac{\int_{N_2 \smallsetminus M_2} \|\operatorname{grad} g\|^2 \tilde{\varphi}^2}{\int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} &= \frac{\int_{\partial M_2}\int_0^{+\infty} \|\operatorname{grad} g\|^2 \tilde{\varphi}^2}{\int_{\partial M_2}\int_0^{+\infty} g^2 \tilde{\varphi}^2} \\ &= \frac{\int_{\partial M_2} \|\operatorname{grad} g\|^2}{\int_{\partial M_2} g^2} + \frac{\int_0^{+\infty} (h')^2 \phi^2}{\int_0^{\infty} h^2 \phi^2}, \end{aligned} \tag{4.5}$$

where we used that in the glued ends $\partial M_2 \times [0,+\infty)$, we have $\tilde{\varphi}(x,t) = \phi(t)$.

Proposition 4.11. *The renormalized operator $S_{\tilde{\varphi}}$ satisfies $\lambda_0(S_{\tilde{\varphi}}) = 0$, which yields that $\lambda_0(S_2) = \lambda_0(S_1)$.*

Proof: Let $\varepsilon > 0$. From Lemma 4.8, there exists $g \in \mathrm{Lip}_c(M_2)$, smooth on $M_2 \smallsetminus C_{t_0}$, for one $t_0 \in (0,\delta)$, not vanishing identically on the boundary, such that

$$\frac{\int_{M_2} \|\operatorname{grad} g\|^2}{\int_{M_2} g^2} < \frac{\varepsilon}{2} \text{ and } \frac{\int_{\partial M_2} \|\operatorname{grad} g\|^2}{\int_{\partial M_2} g^2} < \frac{\varepsilon}{2}.$$

Let $T > 1$ and consider a compactly supported, smooth $h\colon [0,+\infty) \to \mathbb{R}$, with $h(t) = 1$ for $t \leq T$, $h(t) = 0$ for $t \geq T+1$, and $|h'| \leq 2$. Extend $g \in \mathrm{Lip}_c(M_2)$ to the compactly supported $g \in \mathrm{Lip}_c(N_2)$ as in (4.4). Evidently, we have

$$\frac{\int_0^{+\infty} (h')^2\phi^2}{\int_0^{+\infty} h^2\phi^2} \leq 4\frac{\int_T^{T+1} e^{-2t}dt}{\int_1^T e^{-2t}dt} = 4\frac{1-e^2}{e^2-e^{2T}} < \frac{\varepsilon}{2},$$

for some sufficiently large T. From (4.5), it follows that

$$\frac{\int_{N_2 \smallsetminus M_2} \| \operatorname{grad} g \|^2 \tilde{\varphi}^2}{\int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} < \varepsilon.$$

Hence, we have

$$\begin{aligned} \mathcal{R}_{S_{\tilde{\varphi}}}(g) &= \frac{\int_{M_2} \| \operatorname{grad} g \|^2 + \int_{N_2 \smallsetminus M_2} \| \operatorname{grad} g \|^2 \tilde{\varphi}^2}{\int_{M_2} g^2 + \int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} \\ &\leq \max \left\{ \frac{\int_{M_2} \| \operatorname{grad} g \|^2}{\int_{M_2} g^2}, \frac{\int_{N_2 \smallsetminus M_2} \| \operatorname{grad} g \|^2 \tilde{\varphi}^2}{\int_{N_2 \smallsetminus M_2} g^2 \tilde{\varphi}^2} \right\} < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, from Proposition 2.16, it follows that $\lambda_0(S_{\tilde{\varphi}}) = 0$ and in particular, $\lambda_0(S_2) = \lambda_0(S_1)$. ∎

Proof of Theorem 4.5: Consider a Riemannian metric on M_1 as in Lemma 4.7 and its lift on M_2. Glue cylinders along the boundaries and extend the Riemannian covering $p\colon M_2 \to M_1$ to a Riemannian covering $p\colon N_2 \to N_1$ as above. From Remark 4.9, N_1 is complete, without boundary, and with Ricci curvature bounded from below. Consider the Schrödinger operator $S_1 = \Delta + V$ on N_1, as above, and its lift S_2 on N_2. From Remark 4.9, we have that V and $\operatorname{grad} V$ are bounded. From Lemma 4.10 and Proposition 4.11, we obtain that $\lambda_0(S_2) = \lambda_0(S_1) = 0$, and Remark 4.9 yields that $\lambda_0^{\text{ess}}(S_1) \geq 1$. From Theorem 4.2, it follows that the covering $p\colon N_2 \to N_1$ is amenable, and so is the covering $p\colon M_2 \to M_1$. ∎

4.3 Arbitrary Riemannian coverings

In this section, we prove Theorem 4.1 and present some immediate consequences of it. As stated in the Introduction, we establish the following more general version of this theorem, involving manifolds with possibly non-empty boundary.

Theorem 4.12. *Let $p\colon M_2 \to M_1$ be a Riemannian covering. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0^N(S_1) \notin \sigma_{\text{ess}}^N(S_1)$, and S_2 its lift on M_2. Then $\lambda_0^N(S_2) = \lambda_0^N(S_1)$ if and only if the covering is amenable.*

The following lemma, which is a consequence of Theorem 4.5, is essential for the proof of this theorem.

Lemma 4.13. *Let $p\colon M_2 \to M_1$ be a non-amenable Riemannian covering. Let S_1 be a Schrödinger operator on M_1, with $\lambda_0^N(S_1)$ being an eigenvalue of S_1^N, and S_2 its lift on M_2. If $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, then there exists a compact $K \subset M_1$ with non-empty interior, and $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, $\operatorname{supp} f_n \cap p^{-1}(K) = \emptyset$, for any $n \in \mathbb{N}$, and $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$.*

Proof: If M_1 has non-empty boundary, then we denote by ν_i the inward pointing normal to ∂M_i, $i = 1, 2$. From Proposition 1.10, since $p\colon M_2 \to M_1$ is non-amenable, there exists a smoothly bounded, compact domain K', with non-empty interior, such that the covering $p\colon p^{-1}(K') \to K'$ is non-amenable, where $p^{-1}(K')$ may be non-connected. From Theorem 4.5, it follows that $\lambda_0^N(p^{-1}(K')) > 0$.

Since $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N, from Proposition 2.10, there exists a positive function $\varphi \in C^\infty(M_1)$, with $S_1\varphi = \lambda_0^N(S_1)\varphi$ and $\nu_1(\varphi) = 0$ on ∂M_1 (if non-empty). Consider the lift $\tilde{\varphi}$ of φ on M_2 and the renormalization $S_{\tilde{\varphi}}$ of S_2 with respect to $\tilde{\varphi}$. Since $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, from Propositions 2.15 and 2.16, it follows that

$$0 = \lambda_0(S_{\tilde{\varphi}}) = \inf_f \frac{\int_{M_2} \| \operatorname{grad} f\|^2 \tilde{\varphi}^2}{\int_{M_2} f^2 \tilde{\varphi}^2},$$

where the infimum is taken over all non-zero $f \in C_c^\infty(M_2)$, with $\nu_2(f) = 0$ on ∂M_2 (if non-empty). In particular, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2_{\tilde{\varphi}}(M_2)} = 1$, $\mathcal{R}_{S_{\tilde{\varphi}}}(f_n) \to 0$ and $\nu_2(f_n) = 0$ on ∂M_2 (if non-empty).

Since φ is smooth and positive and K' is compact, there exist $c_1, c_2 > 0$, such that $c_1 \leq \varphi \leq c_2$ in K'. From Proposition 2.4, it follows that

$$\frac{\int_{p^{-1}(K')} \| \operatorname{grad} f\|^2 \tilde{\varphi}^2}{\int_{p^{-1}(K')} f^2 \tilde{\varphi}^2} \geq \frac{c_1^2}{c_2^2} \lambda_0^N(p^{-1}(K')) > 0,$$

for any $f \in C_c^\infty(p^{-1}(K')) \smallsetminus \{0\}$. Since $\|f_n\|_{L^2_{\tilde{\varphi}}(M_2)} = 1$ and $\mathcal{R}_{S_{\tilde{\varphi}}}(f_n) \to 0$, it follows that

$$\int_{p^{-1}(K')} f_n^2 \tilde{\varphi}^2 \to 0 \text{ and } \int_{M_2 \smallsetminus p^{-1}(K')} f_n^2 \tilde{\varphi}^2 \to 1.$$

Let $K \subset M_2^\circ$ be a compact set, with non-empty interior, contained in the interior of K'. Let $\chi \in C_c^\infty(M_1)$, with $\chi = 1$ in a neighborhood of K, and $\operatorname{supp} \chi \subset K' \cap M_2^\circ$. Consider the lift $\tilde{\chi}$ of χ on M_2, and let $g_n := (1 - \tilde{\chi}) f_n \in C_c^\infty(M_2)$. It is clear that if M_1 has non-empty boundary, then $\nu_2(g_n) = 0$ on ∂M_2. Moreover, we have

$$\|g_n\|^2_{L^2_{\tilde{\varphi}}(M_2)} = \int_{p^{-1}(K')} (1 - \tilde{\chi})^2 f_n^2 \tilde{\varphi}^2 + \int_{M_2 \smallsetminus p^{-1}(K')} f_n^2 \tilde{\varphi}^2 \to 1$$

and

$$\begin{aligned} \int_{M_2} \| \operatorname{grad} g_n\|^2 \tilde{\varphi}^2 &\leq 2 \int_{p^{-1}(K')} (f_n^2 \| \operatorname{grad} \tilde{\chi}\|^2 + (1 - \tilde{\chi})^2 \| \operatorname{grad} f_n\|^2) \tilde{\varphi}^2 \\ &+ \int_{M_2 \smallsetminus p^{-1}(K')} \| \operatorname{grad} f_n\|^2 \tilde{\varphi}^2 \to 0. \end{aligned}$$

Therefore, $\mathcal{R}_{S_{\tilde{\varphi}}}(g_n) \to 0$ and $\operatorname{supp} g_n \cap p^{-1}(K) = \emptyset$. We may normalize g_n in $L^2_{\tilde{\varphi}}(M)$, so that $\|g_n\|_{L^2_{\tilde{\varphi}}(M_2)} = 1$, for any $n \in \mathbb{N}$.

Consider $h_n := \tilde{\varphi} g_n \in C^\infty_c(M_2)$. If M_2 has non-empty boundary, since we have $\nu_2(\tilde{\varphi}) = \nu_2(g_n) = 0$, it follows that $\nu_2(h_n) = 0$ on ∂M_2. It is clear that $\|h_n\|_{L^2(M_2)} = \|g_n\|_{L^2_{\tilde{\varphi}}(M_2)} = 1$. Moreover, from the definition of the renormalized Schrödinger operator, we have that

$$\begin{aligned} \mathcal{R}_{S_2}(h_n) &= \langle S_2 h_n, h_n \rangle_{L^2(M_2)} = \langle S_{\tilde{\varphi}} g_n, g_n \rangle_{L^2_{\tilde{\varphi}}(M_2)} + \lambda_0^N(S_2) \\ &= \mathcal{R}_{S_{\tilde{\varphi}}}(g_n) + \lambda_0^N(S_2) \to \lambda_0^N(S_2), \end{aligned}$$

which completes the proof. ∎

Proof of Theorem 4.12: From Corollaries 3.17, 3.18, Proposition 2.5 and Remark 2.6, if the covering is infinite sheeted and amenable, then $\lambda_0^N(S_1) = \lambda_0^N(S_2)$. If the covering is finite sheeted, then for $f \in C^\infty_c(M_1)$, we have that $f \circ p \in C^\infty_c(M_2)$, and the equality of the bottoms follows from Propositions 2.4, 2.5 and Remark 2.6. Hence, it remains to prove the converse implication.

Assume to the contrary that the covering is non-amenable. From Lemma 4.13, since $\lambda_0^N(S_2) = \lambda_0^N(S_1) \notin \sigma^N_{\mathrm{ess}}(S_1)$, it follows that there exists a compact $K \subset M_1$ with non-empty interior, and a sequence $(f_n)_{n\in\mathbb{N}} \subset C^\infty_c(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, $\operatorname{supp} f_n \cap p^{-1}(K) = \emptyset$, for any $n \in \mathbb{N}$, and $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$. For any $n \in \mathbb{N}$, consider the pushdown g_n of f_n, defined by

$$g_n(z) := \Big(\sum_{y \in p^{-1}(z)} f_n(y)^2 \Big)^{1/2},$$

for any $z \in M_1$. Then $g_n \in \operatorname{Lip}_c(M_1)$, $\|g_n\|_{L^2(M_1)} = 1$ and $\mathcal{R}_{S_1}(g_n) \le \mathcal{R}_{S_2}(f_n)$, for any $n \in \mathbb{N}$ (cf. [3, Section 4]). From Proposition 2.4, since $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, it follows that $\mathcal{R}_{S_1}(g_n) \to \lambda_0^N(S_1)$. From Proposition 2.11, since $\lambda_0^N(S_1) \notin \sigma^N_{\mathrm{ess}}(S_1)$ and $\operatorname{supp} g_n \cap K = \emptyset$, this is a contradiction. Hence, the covering is amenable. ∎

Proof of Theorem 4.1: Follows from Theorem 4.12, since the manifolds involved may have empty boundary. ∎

Remark 4.14. In Theorem 4.1, the manifolds do not have to be complete. Therefore, from Remark 2.1, we obtain the corresponding statement for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary.

Corollary 4.15. *Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_1 compact. Then the covering is amenable if and only if it preserves the bottom of the Dirichlet/Neumann spectrum of some/any Schrödinger operator.*

Proof: Follows from Theorem 4.12 and Remark 4.14, since the Dirichlet and the Neumann spectrum of a Schrödinger operator on a compact manifold is discrete. ∎

The next example demonstrates that the assumption $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$ in Theorem 4.1 cannot be replaced with $\lambda_0(S_1)$ being an eigenvalue of the Friedrichs extension of S_1.

Example 4.16. Let M_1 be a two dimensional torus with a cusp attached, endowed with a Riemannian metric, such that M_1 is complete and outside a compact set, the cusp is the surface of revolution generated by $1/t^2$, with $t \geq 1$. Since M_1 has finite volume, it follows that $\lambda_0(M_1) = 0$ and constant functions are $\lambda_0(M_1)$-eigenfunctions of the Friedrichs extension of the Laplacian on M_1. Let x be a point of the torus and consider the non-negative quantity

$$\mu := -\lim_{r\to+\infty} \frac{1}{r} \ln(\mathrm{Vol}(M_1) - \mathrm{Vol}(B(x,r))) \leq -\lim_{r\to+\infty} \frac{1}{r} \ln(2\pi \int_{r+1}^{+\infty} \frac{1}{t^2} dt) = 0.$$

From [7, Theorem 1], it follows that $\lambda_0^{\text{ess}}(M_1) = 0$. Consider the universal covering $p\colon M_2 \to M_1$. Since $\pi_1(M_1)$ is the free group with two generators, it follows that p is non-amenable (from Example 1.14). Since the fundamental group of the cusp is amenable, from Corollary 3.32, it follows that $\lambda_0(M_2) = 0$.

It is clear that Theorem 4.1 is more general than the results of [2,8,21]. For sake of completeness, we present an example demonstrating this fact. Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_1 non-compact, complete, without boundary, and with $\sigma_{\text{ess}}(M_1) = \emptyset$. Then, from Theorem 4.1, it follows that $\lambda_0(M_2) = \lambda_0(M_1)$ if and only if the covering is amenable. It is worth to point out that since we do not require the covering to be normal, the results of [8,21] cannot be applied in this case. Moreover, from [12, Theorem 3.1], it follows that the Ricci curvature of M_1 is not bounded from below. Hence, also the result of [2] cannot be applied in this case.

4.4 An application

The aim of this section is to prove the following proposition, which was established for the Laplacian on manifolds without boundary in [2].

Proposition 4.17. *Let $p\colon M_2 \to M_1$ be an infinite sheeted Riemannian covering. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. If $\lambda_0^N(S_1) = \lambda_0^N(S_2)$, then $\lambda_0^N(S_2) \in \sigma_{\text{ess}}^N(S_2)$.*

The main point of this proposition is that the covering is not required to have infinite deck transformations group, since in this case, according to Corollary 3.27, the spectrum of S_2^N coincides with its essential spectrum. It is worth to point out that the manifolds in this proposition may have empty boundary. Moreover, since they may be non-complete, from Remark 2.1, the analogous statement holds for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary.

Proposition 4.18. *Let $S = \Delta + V$ be a Schrödinger operator on a Riemannian manifold M and consider $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_c(M)$, with $\|f_n\|_{L^2(M)} = 1$ and $\mathcal{R}_S(f_n) \to \lambda_0^N(S)$. If $\lambda_0^N(S)$ is not an eigenvalue of S^N, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$, such that $f_{n_k} \rightharpoonup 0$ in $L^2(M)$.*

Proof: From Proposition 2.3, there exists a sequence $(f_n') \in C_c^\infty(M) \cap \mathcal{D}(S^N)$, with $\|f_n'\|_{L^2(M)} = 1$ and $\|f_n - f_n'\|_{H_V(M)} \leq 1/n$, for any $n \in \mathbb{N}$, where $H_V(M)$ is the space defined in Chapter 2. It is clear that $\mathcal{R}_S(f_n') \to \lambda_0^N(S)$ and it suffices to prove the statement for $(f_n')_{n\in\mathbb{N}}$. From the Spectral Theorem (cf. [24, Chapter 8]), there exists a measure space X, such that $L^2(M)$ is isometrically isomorphic to $L^2(X)$, and under this identification, S^N corresponds to a multiplication operator with a measurable function $f\colon X \to \mathbb{R}$; that is, an operator $\mu_f\colon \mathcal{D}(\mu_f) \subset L^2(X) \to L^2(X)$, with $\mathcal{D}(\mu_f) := \{g \in L^2(X) : fg \in L^2(X)\}$ and $\mu_f(g) = fg$, for any $g \in \mathcal{D}(\mu_f)$. The spectrum of S^N coincides with the essential range of f and in particular, $f \geq \lambda_0^N(S)$ almost everywhere.

Let $(g_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mu_f)$ be the sequence corresponding to $(f_n')_{n\in\mathbb{N}}$ under this identification. Since $\|g_n\|_{L^2(X)} = 1$, after passing to a subsequence, we have that $g_n \rightharpoonup g$, for some $g \in L^2(X)$. It is clear that

$$\int_X (f - \lambda_0^N(S)) g_n^2 = \langle \mu_f g_n, g_n \rangle_{L^2(X)} - \lambda_0^N(S) = \mathcal{R}_S(f_n') - \lambda_0^N(S) \to 0.$$

For $\varepsilon > 0$, consider the measurable set $A_\varepsilon := \{f \geq \lambda_0^N(S) + \varepsilon\}$. Evidently, we have

$$\int_{A_\varepsilon} g_n^2 \leq \frac{1}{\varepsilon} \int_{A_\varepsilon} (f - \lambda_0(S^N)) g_n^2 \to 0.$$

Since $g_n \rightharpoonup g$ in $L^2(X)$, this yields that $g = 0$ almost everywhere in A_ε. In particular, $g = 0$ almost everywhere in $X \smallsetminus f^{-1}(\{\lambda_0^N(S)\})$, which yields that $\mu_f g = \lambda_0^N(S) g$. Since $\lambda_0^N(S)$ is not an eigenvalue of S^N, it follows that $g = 0$. Therefore, $g_n \rightharpoonup 0$ in $L^2(X)$, which yields that $f_n' \rightharpoonup 0$ in $L^2(M)$. ∎

Lemma 4.19. *Let $p\colon M_2 \to M_1$ be a Riemannian covering. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. If $\lambda_0^N(S_2) = \lambda_0^N(S_1) \notin \sigma_{\mathrm{ess}}^N(S_2)$, then $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N.*

Proof: Assume to the contrary that $\lambda_0^N(S_1)$ is not an eigenvalue of $\lambda_0^N(S_1)$. From Proposition 2.10, there exists a square-integrable, $\lambda_0^N(S_2)$-eigenfunction φ of S_2^N, which is smooth and positive in M_2. Without loss of generality, we may assume that $\|\varphi\|_{L^2(M_2)} = 1$. Since $\varphi \in H_{V\circ p}(M_2)$, there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$ and $f_n \to \varphi$ in $H_{V\circ p}(M_2)$, where $H_{V\circ p}(M_2)$ is the space defined in Chapter 2. Evidently, we have that $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$.

Consider the pushdowns

$$g_n(z) = \Big(\sum_{y\in p^{-1}(z)} f_n(y)^2\Big)^{1/2}.$$

on M_1, with $n \in \mathbb{N}$. Then $g_n \in \mathrm{Lip}_c(M_1)$, $\|g_n\|_{L^2(M_1)} = 1$ and $\mathcal{R}_{S_1}(g_n) \leq \mathcal{R}_{S_2}(f_n)$, for any $n \in \mathbb{N}$ (cf. [3, Section 4]). From Proposition 2.4, since $\lambda_0^N(S_1) = \lambda_0^N(S_2)$, it follows that $\mathcal{R}_{S_1}(g_n) \to \lambda_0^N(S_1)$. Since $\lambda_0^N(S_1)$ is not an eigenvalue of S_1^N, from Proposition 4.18, after passing to a subsequence, we have that $g_n \rightharpoonup 0$ in $L^2(M_1)$.

Consider a non-negative $\chi_2 \in C_c^\infty(M_2) \smallsetminus \{0\}$, and its pushdown $\chi_1 \in \mathrm{Lip}_c(M_1)$ on M_1. Then

$$\begin{aligned} \langle \chi_2, f_n\rangle_{L^2(M_2)} &= \int_{M_1} \sum_{y\in p^{-1}(z)} \chi_2(y) f_n(y)\,dz \\ &\leq \int_{M_1} \Big(\sum_{y\in p^{-1}(z)} \chi_2(y)^2\Big)^{1/2} \Big(\sum_{y\in p^{-1}(z)} f_n(y)^2\Big)^{1/2} dz \\ &= \langle \chi_1, g_n\rangle_{L^2(M_1)}. \end{aligned}$$

This is a contradiction, since $\langle \chi_1, g_n\rangle_{L^2(M_1)} \to 0$ and $\langle \chi_2, f_n\rangle_{L^2(M_2)} \to \int_{M_2} \chi_2 \varphi > 0$. Therefore, $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N. ∎

Proof of Proposition 4.17: If the covering is amenable, then the claim follows from Corollaries 3.17 and 3.18. Hence, it remains to prove the statement for p non-amenable. Assume to the contrary that $\lambda_0^N(S_2) \notin \sigma_{\mathrm{ess}}^N(S_2)$. From Lemma 4.19, it follows that $\lambda_0^N(S_1)$ is an eigenvalue of S_1^N. Since $\lambda_0^N(S_2) = \lambda_0^N(S_1)$, from Lemma 4.13, there exists a compact subset K of M_1 with non-empty interior, and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2) \smallsetminus \{0\}$, with $\operatorname{supp} f_n \cap p^{-1}(K) = \emptyset$, for any $n \in \mathbb{N}$, and $\mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2)$. From Proposition 2.11, since $\lambda_0^N(S_2) \notin \sigma_{\mathrm{ess}}^N(S_2)$ and $p^{-1}(K)$ contains compact sets of positive measure, this is a contradiction. ∎

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